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Talk Title: Newton-Okounkov bodies and cluster duality for Grassmanians

Date and Time: January 16th, 2018 - 2pm

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arXiv: 1712.00447

Starting point for work

Gelfand-Tsetlin polytopes can be descirbed in 2 dual ways:

1. Lattice points index basis of highest weight representations V_λ.

2. Their inequalities come from tropicalizing geometric crystals.

Definition: Grassmannian $Gr_k(\mathbb{C}^n) = \{V \subset \mathbb{C}^n \mid \dim V = k\}$

Represent elements by $k \times n$ matrices M.

For $I \in \binom{[n]}{k}$, $P_I(M)$ = det of $k \times k$ minor of M in columns I.

↑ Plücker coordinates

Overview

Grassmannian

$X = Gr_{n-k}(\mathbb{C}^n)$

Fix ample divisor

$D = \{P_{\{1,2,\dots,n-k\}} = 0\} \subset X$

Plucker coords

P_J for $J \in \binom{[n]}{n-k}$

Mirror

$(\check{X}, W)$ - LG model

$\check{X}$ = complement of anticanonical divisor in $Gr_k(\mathbb{C}^n)$

Remove locus where any cyclically consecutive Plücker vanishes

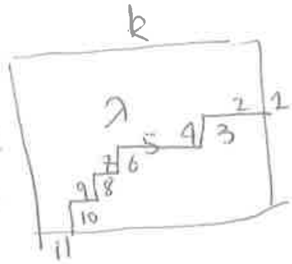
$W: \check{X} \to \mathbb{C}(q)$

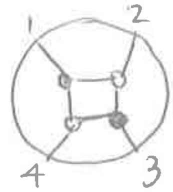
Superpotential (Marsh-Rietsch)

$N := k(n-k) = \dim X = \dim \check{X}$

Plucker coords P_I for $I \in \binom{[n]}{k}$

Prefer: Index Plucker coords by Young diagrams

J ⟷ vertical steps ⟵  ⟼ I ⟷ horizontal steps.

Fix reduced plabic graph G of type (k,n) 

Many of these: each determines seed for cluster variety.

Cluster ($\mathcal{X}$ or $\mathcal{A}$)-varieties = varieties covered by tori w/ nice transition maps.

"Network chart"

$\Phi_G^{\mathcal{X}} : (\mathbb{C}^*)^N \to X$

Newton-Okounkov body Δ_G

NObody = moment polytope

arbitrary variety — toric variety

Δ_G = polytope defined as convex hull

Lattice points $r\Delta_G \leftrightarrow$ basis of $H^0(X, \mathcal{O}(rD))$ $\cong V_{r\omega_k}$

Cluster chart

$\Phi_G^{\mathcal{A}} : (\mathbb{C}^*)^N \to \check{X}$

Write superpotential W in terms of $\Phi_G^{\mathcal{A}}$, & "tropicalize" it:

⇝

Q_G = polytope defined by inequalities.

Theorem 1 (Rietsch-W): $\Delta_G = Q_G$

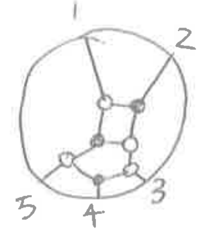
Theorem 2 (") combinatorial formula for lattice points

Definition (Postnikov):

Plabic graph is planar graph G in disk with n bdry vertices 1,2,..,n.

Each bdy vertex incident to ! edge

Internal vertices are ● ○

Ex: 

Def/Lem: Turn right at ●
left at ○

The trip T_i = path starting at i and following rules to end at bdy vertex $\pi_G(i)$

$$\pi_G = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 5 & 1 & 2 \end{matrix}$$

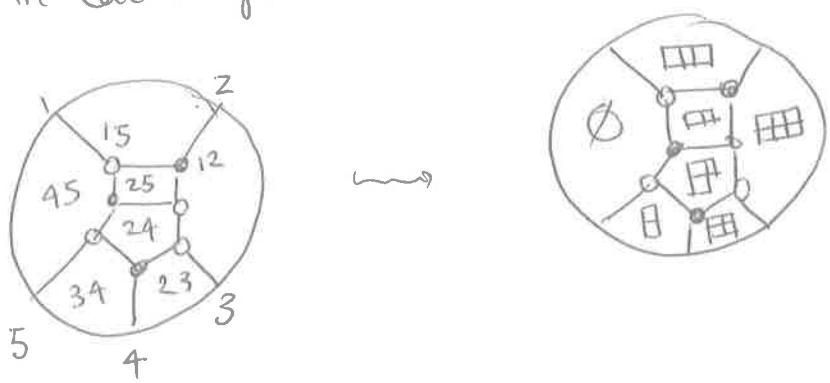
G has type (k,n) if it has n bdy vertices, $k(n-k)+1$ regions

and $$\pi_G = \begin{matrix} 1 & 2 & \cdots & k & k+1 & \cdots & n \\ \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ n-k+1 & n-k+2 & & n & 1 & & n-k \end{matrix}$$

Given G use trips T_i to label faces by partitions $\subseteq (n-k) \times k$.

T_i divides disk into 2 parts (L, R)

Put i in each region to L.



$k=3$
$n=5$

G of type (k,n)

Postnikov → Network chart $\Phi_G^X : (\mathbb{C}^*)^N \to X$

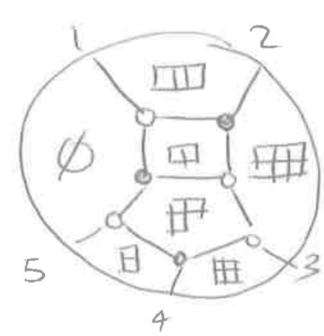
Scott → Cluster chart $\Phi_G^A : (\mathbb{C}^*)^N \to \check{X}$.

- Network chart

Put var x_μ in region labeled μ

choose acyclic perfect orientation $\mathcal{O}$ of G w/ source $I = \{1, \ldots, n-k\}$

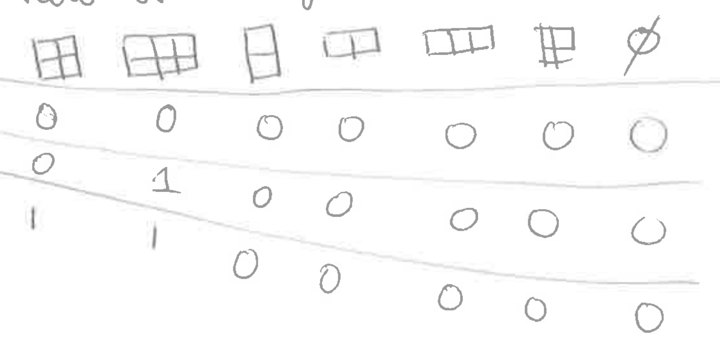
- each ● has ! outgoing edge
  " ○ " " ingoing edge



$\Phi_G^{\chi} : (\mathbb{C}^*)^N \longrightarrow Gr_{n-k}(\mathbb{C}^n)$ Def/Thm: This map is injective/well-defined.

$\{x_\mu\} \mapsto \Phi_G^{\chi}(\{x,\mu\})$

$P_J(\Phi_G^{\chi}\{x,\mu\})$ = gen. function for non-intersecting paths $I \to J$.

Plucker coord's $\xrightarrow{val_G}$	Lattice points which is the exponent vector of leading term of P_J
$P_{12} = 1$	
$P_{13} = x_{\boxplus}$	
$P_{14} = x_{\boxplus} x_{\boxplus} + x_{\boxplus} x_{\boxplus} x_{\boxplus}$	
$\vdots$	

Recall $p := \{P_{12 \ldots n-k} = 0\} \subset X$

Def: Let $\mathcal{L}_r = H^0(X, \mathcal{O}(rD))$ = {degree r polynomials in Plücker coords}

The Newton-Okounkov body is

$$\Delta_G := \text{Conv. Hull}\left(\bigcup_{r=1}^{\infty} \tfrac{1}{r} \, val_G(\mathcal{L}_r \setminus \{0\})\right)$$

Thm2 (Rietsch-W)
Lattice points of $\Delta_G(D)$ are precisely valuations of the Plucker coords $val_G(p_\lambda) \subset \mathbb{R}^N$. Have formula:

$val_G(p_\lambda) \subseteq \mathbb{Z}^{\text{faces}(G)}$

For $\mu \in \text{face}(G)$, the μth coordinate of $val_G(p_\lambda)$ is

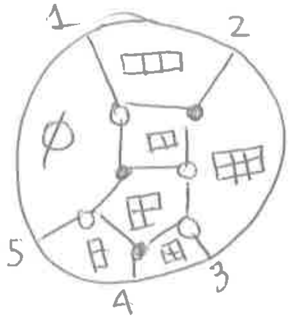
$$(val_G(p_\lambda))_\mu = \text{MaxDiag}(\mu \setminus \lambda)$$

Ex: $val_G(P_{\boxplus})_{\boxplus} = \text{MaxDiag}(\boxplus \setminus \boxplus) = 1$

$val_G(P_{\square})_{\boxplus} = \text{MaxDiag}(\boxplus \setminus \square) = 2$

Remark: $\text{MaxDiag}(\mu \setminus \lambda)$ = minimal diagonal st q^d appears in schubert expansion of $\sigma_\mu * \sigma_{\lambda^c}$ in $QH^*(Gr_{n-k}(\mathbb{C}^n))$

Mirror



Plucker coords

$\{P_\emptyset, P_{(3)}, P_{(2)}, P_{(2,1)}, P_{(3,3)}$, [illegible] $P_{(1,1)}, P_{(1,1,1)}\}$ form cluster

for $\mathbb{C}[Gr_k(\mathbb{C}^n)]$

ie every plucker coord can be written as a pos. Laurent poly in these

Superpotential: $W: \check{X} \to \mathbb{C}(q)$

Ex: $\check{X} \subset Gr_3(\mathbb{C}^5)$

$$W = \frac{P_{\square}}{P_\emptyset} + q\frac{P_{(2)}}{P_{(3,3)}} + \frac{P_{(3,1)}}{P_{(3)}} + \frac{P_{(3,2)}}{P_{(3,3)}} + \frac{P_{(2,2)}}{P_{(1,1)}}$$

Rewrite W in terms of cluster(G)

$$W = \frac{P_{(2,2)}}{P_{(2,1)}} + \frac{P_{(1,1)} + P_{(2)}}{P_{(2,1)}} + q\frac{P_{(2)}}{P_{(3,3)}} + \cdots$$

"Tropicalize" W: $b_{(2,2)} - b_{(2,1)} \geq 0 \;|\; b_{(1,1)} + b_{(2)} - b_{(2,1)} \geq 0 \;|\; 1 + b_{(2)} - b_{(3,3)} \geq 0$

Defines polytope Q_G

Theorem 1: $\Delta_G = Q_G$

Ex: $G =$