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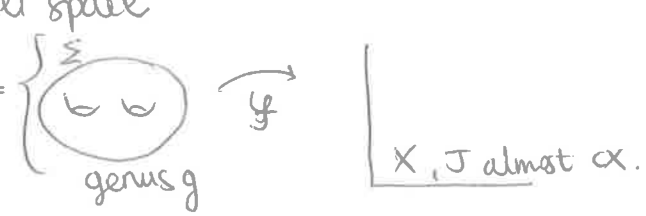
Talk Title: The Gopakumar-Vafa conjecture for symplectic manifolds.

Date & Time: January 19, 2018 - 3:30pm

Assume X symplectic CY 3-fold, i.e. $c_1(X)=0$ $\dim_{\mathbb{C}} X = 3$.

Consider moduli space

$\overline{\mathcal{M}}_{A,g}(X) = \Big\{$



f J-holom. $\bar\partial_J f = 0$

$f_*[\Sigma] = A \in H_2(X)$ $\Big\} / \sim$ parametriz.

If X sympl. CY 3-fold

$$\dim \overline{\mathcal{M}}_{A,g}(X) = 0, \ \forall g, A$$

"Expect" fin. many curves

{count them with sign and $\frac{1}{\#\mathrm{Aut}}$} $\to GW_{A,g}(X) \in \mathbb{Q}$

problem: multiple covers. ← come in families.

$$GW_{A,g}(X) = \int_{[\overline{\mathcal{M}}_{A,g}(X)]^{vir}} 1 \in \mathbb{Q}$$

Gopakumar Vafa Conj

For a complex CY 3-fold X

$$GW(X) = \sum_{\substack{A \neq 0 \\ g}} GW_{A,g}\, t^{2g-2} q^A \overset{(*)}{=} \sum_{\substack{A\neq 0 \\ g}} n_{A,g}(X) \sum_{k=1}^{\infty} \frac{1}{k}\left(2\sin\frac{kt}{2}\right)^{2g-2} q^{kA}$$

where $n_{A,g}(X)$ (BPS numbers) satisfy

a) $n_{A,g}(X) \in \mathbb{Z}$

b) fix A, $n_{A,g}(X) = 0$ for $g >> 0$.

(*) $\{GW\} \underset{\text{GV transform}}{\longleftrightarrow} \{n\}$

In $4\dim_{\mathbb{R}}$ GW $\longleftrightarrow$ SW $\longleftrightarrow$ Don

Idea: I. Prove a local version of the conjecture for special local CY

$\leftarrow$ special alm cpx on normal bundle

II. Reduce it to the local conjecture

Structure Theorem (I-Parker)

For any symplectic Calabi-Yau 3-fold $\exists!$ elem. invar. $e_{A,g}(X) \in \mathbb{Z}$ s.t.

$$GW(X) = \sum_{\substack{A \neq 0 \\ g}} e_{A,g}(X) \cdot G_g(t, q^A)$$

$e_{A,g}(X) \leftarrow$ virtual count

where G_g = universal generating function,

- counts contribution of multiple covers of genus g "elem. curve"

$= GW^{loc}(C)$

Eleme Curves? need to be able to calculate contribution of multiple covers.

$g = 0$ $\quad \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$

higher genus: start $L \oplus L \to C$, $L^{\otimes 2} = K_C$

deform $J_0 \rightsquigarrow J_\alpha$ almost cx structure using "double" construction due Junho Lee

$\Rightarrow$ • C super-rigid: all multiple covers have no 1st order deformations off C.

$$GW^{loc}(C) = \int_{[\mathcal{M}(C)]^{vir}} e(Ob)$$

$\uparrow$ moduli space of covers

Ob = obstruction bundle

actual bundle.

• $Ob = Ob^1 \oplus Ob^1$

$$e(Ob) = \pm e_{top}(Ob^1 \otimes_{\mathbb{R}} \mathbb{C}) = \pm c_{top}(-\mathrm{ind}\,\bar{\partial}_L \otimes_{\mathbb{R}} \mathbb{C})$$

$$\mathrm{ind}\,\bar{\partial}_L = R^\bullet \pi_* L$$

<u>Theorem</u> (I-Parker)

The local contribution of multiple covers of an elementary curve.

$$\exp(G_g) = 1 + \sum_{d=1}^{\infty} \sum_{\rho \vdash d} \left(\prod_{\square \in \rho} 2\sin\frac{h(\square)t}{2} \right)^{2g-2} q^d$$

where ρ partition of d $\rightsquigarrow$

$\updownarrow$

ρ ind. representation of S_d

$h(\square)$ = hook length of $\square$

R.H.S has been computed by Bryan-Pandharipande using TQFT.

<u>Note</u>: Leading coefficient in t (in each degree q^d)

LHS $\rightsquigarrow$ counts unramified degree d covers of C

$\{\pi_1(C) \to S_d\}/\mathrm{conj}$

RHS $\rightsquigarrow \displaystyle\sum_{\substack{\rho \text{ irred.} \\ \text{rep. } S_d}} (\dim \rho)^{2-2g}$

<u>Prop</u>: Local GV conjecture for "elementary local model"

i.e. $\{G_g\} \xleftrightarrow{\text{GV transform}} \{n\}$ ← local BPS states

satisfy ⓐ + ⓑ

<u>e.g.</u> $g=0$ $\quad G_g = \sum_{k=1}^{\infty} \frac{1}{k}\left(2\sin\frac{kt}{2}\right)^{-2} q^k$

so ⇒ 1 BPS state in degree 1 and $g=0$

<u>e.g. $g=1$</u> 1 BPS state in $\forall$ deg ≥ 1 and $g=1$

<u>Corollary</u>: $\forall$ symplectic CY 3fold, $n_{A,g}(X) \in \mathbb{Z}$

Finiteness still open, reduced to finiteness of $e_{A,g}(X)$

Note: Proved more general ~~formula~~ structure theorem + GV conj for $\forall$ symplectic 3-fold.

Extends to Real GW invariants of Real symplectic ~~C~~Y 3-folds
(w/ P. Georgieva)

X Real i.e. involution c_X antisymplectic $c_X^* \omega = -\omega$
$c_X^* J = -J$

count Real maps
$f \circ c_\Sigma = c_X \circ f$

- obtain similar structure theorem for Real GW
- Calculate $R_g \leftarrow$ contributions of Real multiple covers of $\mathbb{C}$

(work in progress)

$\rightsquigarrow$ Klein TQFT

extends BP TQFT

- deformation, splitting

GV conj. for Real CY 3 fold

$$\sum_{\substack{A \neq 0 \\ g}} RGW_{A,g}(X) t^{g-1} q^{A/2} = \sum_{\substack{A \neq 0 \\ g}} n_{A,g}(X) \sum_{k \text{ odd}} \frac{1}{k} \left(2 \sin \frac{kt}{2}\right)^{g-1} q^{kA}$$

extends earlier work.

- Katz-Liu calc.
- A-Zarnotaev: restrict to $g=0$ and A odd

$$\sum_{A \text{ odd}} RGW_{A,0}\, q^{A/2} = \sum_{A \text{ odd}} \sum_{k \text{ odd}} \frac{1}{k^2} q^{k/2}$$

Structure Theorem:

decompose $\overline{\mathcal{M}}(X) = \bigsqcup_i \mathcal{O}_i$ open + closed subsets
clusters of curves.

$$GW(X) = \sum_i GW^{loc}(\mathcal{O}_i)$$