MODULI SPACES OF STABLE MAPS AND GW THEORY, I

RAHUL PANDHARIPANDE*

(*) Notes taken by Dhyan Aranha, all errors should be attributed to me and my ignorance about the subject. Corrections and suggestions are welcome, and should be sent to: dhyan.aranha@gmail.com.

We start here with X be a non-singular projective variety over $\mathbb{C}$. The beginning of this subject at least from the point of view of this talk and actually many points of view is the definition of the moduli space of stable maps

$$\overline{\mathcal{M}}_{g,n}(X,\beta)$$

g - genus,
n - number of marked points,
$\beta \in H_2(X,\mathbb{Z})$.

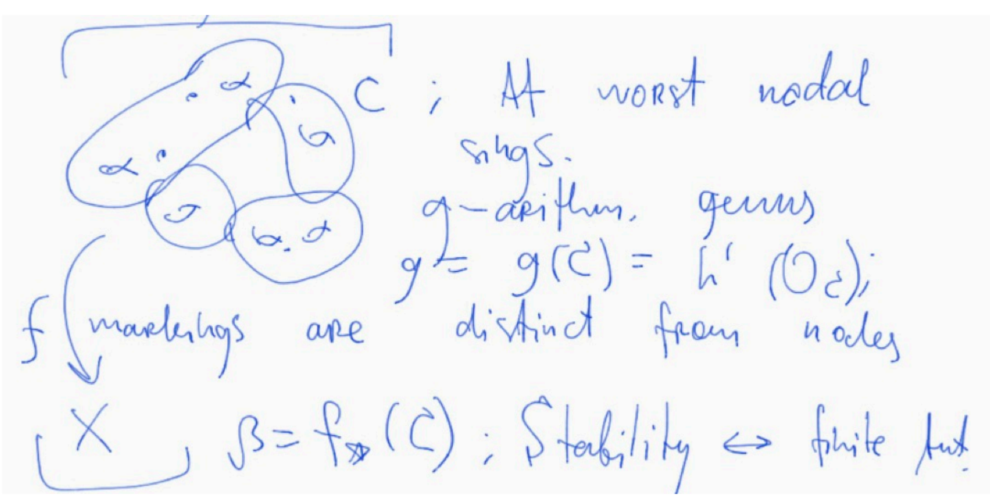
Roughly, a point in this space is an algebraic morphism

$$f : C \longrightarrow X$$

from a connected, reduced, etc.. curve C with at worst nodal singularities, whose genus is $g = g(C) = h^1(\mathcal{O}_C)$, whose markings are distinct and are away from the nodes, and has $f_*[C] = \beta$. Finally there is a stability condition: finite automorphisms (i.e. automorphisms of the domain that

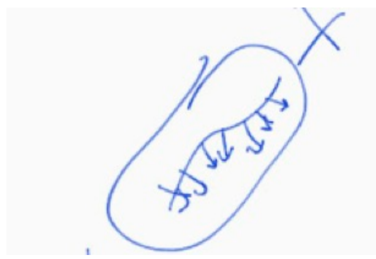
The proper point of view on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is that it is a Deligne-Mumford stack furthermore one can show:

- $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is proper (not really difficult)



• $\overline{\mathcal{M}}_{g,n}(X,\beta)$ carries a virtual fundamental class (not really easy, Li-Tian, Behrend-Fantechi)

Now suppose you have a curve C and an algebraic morphism $f : C \longrightarrow X$. You can deform C and you can deform the map f. Let's for a keep C fixed and deform f:



$$\begin{aligned} \mathrm{Def}(f) &= H^0(C, f^*T_X) \\ \mathrm{Obs}(f) &= H^1(C, f^*T_X) \end{aligned}$$

$$\text{Vir } \dim_{\mathbb{C}} = 3g - 3 + n + \chi(C, f^*T_X)$$

we can rewrite this as

$$\text{Vir } \dim_{\mathbb{C}} = \int_\beta c_1(X) + (\dim_{\mathbb{C}}(X) - 3)(1-g) + n$$

The virtual fundamental class is a cycle in $A_{\text{Vir } \dim_{\mathbb{C}}}(\overline{\mathcal{M}}_{g,n}(X,\beta))$, that is

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}} \in A_{\text{Vir } \dim_{\mathbb{C}}}(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

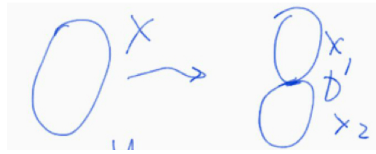
Now we can make a sort of flow chart of things:

(i) Is $\overline{\mathcal{M}}_{g,n}(X,\beta)$ pure (e.g. all irreducible components have expected dimension) of expected dimension? $\rightsquigarrow$ $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}$ is just the fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]$.

(ii) Is $\overline{\mathcal{M}}_{g,n}(X,\beta)$ non singular irreducible of wrong dimension (i.e. ¿ Vir $\dim_{\mathbb{C}}$)? $\rightsquigarrow$ $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}$ is $c_{top}(\text{Obs})$.

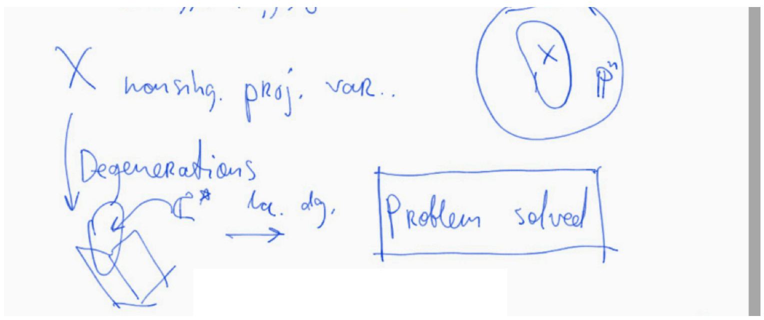
(iii) Does X have a torus-action: $\mathbb{C}^* \curvearrowright X$. $\rightsquigarrow$ There is a localization formula due to Graber-Pandharipande, that says roughly you can write $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}$ in terms of the virtual classes on the $\mathbb{C}^*$-fixed loci of $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

(iv) Lastly you can try to deform X. $\rightsquigarrow$ Breaking:



The virtual class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ can be expressed in terms of $[\overline{\mathcal{M}}_{g_1,n_1}(X_1/D,\beta_1)]^{\mathrm{vir}}$ and $[\overline{\mathcal{M}}_{g_2,n_2}(X_2/D,\beta_2)]^{\mathrm{vir}}$. This idea is contained in the subject of relative Gromov-Witten theory, due to Li-Ruan, Li, Ionel-Parker, etc...

So in general if you want to know what the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ is say for some smooth projective variety X. You might consider the following diagram



Log Gromov-Witten theory: Gross-Siebert, Abramovich-Chen-Gross-Siebert.

Examples: Consider $\overline{\mathcal{M}}_{1,1}(X,0) = \overline{\mathcal{M}}_{1,1} \times X$, with $d := \dim_{\mathbb{C}} X$. Since $\dim \overline{\mathcal{M}}_{1,1} = 1$, the space $\overline{\mathcal{M}}_{1,1}(X,0)$ has dimension $d+1$. Points in this moduli space look like

What is the virtual dimension? Well we use our formula and get Vir $\dim_{\mathbb{C}} = 0+0+1 = 1$.

So now where are we in our flow chart? We aren't in case (i) because if d is positive the dimension is unexpected but we are in case (ii), non-singular but of the wrong dimension. This means we have an obstruction bundle on $\overline{\mathcal{M}}_{1,1} \times X$ which comes from the d-dimensional vector space $H^1(C, f^*T_X)$. We see that the obstruction bundle is

$$\mathrm{Obs} = \mathbb{E}^{\vee} \otimes T_X$$

Remark 0.0.1. *Sometimes people write the tensor product appearing in the formula above as $\boxtimes$ since it is the exterior tensor product.*

Where $\mathbb{E}$ is the hodge bundle over $\overline{\mathcal{M}}_{1,1}$. The Hodge bundle exists for any $\overline{\mathcal{M}}_{g,n}$ and can be described fiber-wise as: Given a point in the moduli space $[C, p_1, \cdots, p_n]$ the fiber is $H^0(C, \omega_C)$.

Now since we are looking at genus 1 curves the rank of $\mathbb{E}^\vee$ will be 1. Also since X is d-dimensional we see that T_X has rank d so their tensor product will also be rank d. Thus the virtual class is

$$[\overline{\mathcal{M}}_{1,1}(X,0)]^{\text{vir}} = c_d(T_X) - \lambda c_{d-1}(T_X)$$

where $\lambda = c_1(\mathbb{E})$.

It turns out there's more to say about this. A GW invariant is a pairing against this virtual class a particular kind:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}(X,\beta) & \xrightarrow[ev_1,\ldots,ev_n]{\vdots} & X \\ \downarrow{\scriptstyle forget} & & \\ \overline{\mathcal{M}}_{g,n}(X,\beta) & & \end{array}$$

The maps ev_i are just the evaluation on the marked point. The morphism "forget" we only have when $2g - 2 + n > 0$.

If you accept the basic claim that this subject is about the exploration of the virtual class. The first question you could ask is: How could you possibly tell me what the virtual class is?

One of the things you could do is consider the morphism

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \xrightarrow{j} \overline{\mathcal{M}}_{g,n} \times X^{\times n}$$

and look at $j_*[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}} \in H_*(\overline{\mathcal{M}}_{g,n} \times X^{\times n})$. This has the advantage that you have a Künneth formula.

Given $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{C})$, the Gromov-Witten invariants are

$$\langle \gamma_1, \ldots, \gamma_n \rangle^X_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \in \mathbb{C}.$$

there are fancier Gromov-Witten invariants, the so-called "descendent" invariants

$$\langle \tau_{k_1}(\gamma_1) \ldots, \tau_{k_n}(\gamma_n) \rangle^X_{\ldots} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}} \psi_1^{k_1} ev_1^*(\gamma_1) \cup \cdots \cup \psi_n^{k_n} ev_n^*(\gamma_n)$$

where the ψ_i's are the cotangent line classes.

How might we get some numbers out of the virtual class. Consider again $\overline{\mathcal{M}}_{1,1}(X,0)$:

$$\begin{aligned}\langle\, c_1(X)\rangle_{1,1} &= \int_{\overline{\mathcal{M}}_{1,1}\times X}(c_d(X)-\lambda c_{d-1}(X))\cdot c_1(X)\\ &= -\frac{1}{24}\cdot\int_X c_d(X)c_1(X)\end{aligned}$$

The chern number $\int_X c_d(X)c_1(X)$ is somewhat magical, we'll come back to it.

What's the other thing you can do? Well you could pull back a point class via

$$\begin{array}{c}\overline{\mathcal{M}}_{1,1}(X,0)\\ \big\downarrow\\ \overline{\mathcal{M}}_{1,1}.\end{array}$$

In other words you can fix a particular complex structure and write

$$\overline{\mathcal{M}}_{E,1}(X,0)$$

for E an elliptic curve. (This will also cut the dimension by 1) We get a number

$$\langle 1\rangle_{E,1}=\int_X c_d(X)$$

the Chern number $\int_X c_d(X)$ is just the Euler characteristic (determined by the Betti # 's). You can ask which Chern numbers are determined by the Betti numbers, the answer it turns out is only $\int_X c_d(X)$. So then you might as which Chern numbers are determined by the Hodge numbers, and the answer is that there are two: $\int_X c_d(X)$ and $\int_X c_d(X)c_1(X)$. This is actually very important, it has to do with how to write the Virasoro constraints.

Another example with an elliptic curve: Let $X=E$, we can consider

$$\overline{\mathcal{M}}_1(E,d)$$

which has Vir $\dim_{\mathbb{C}}=0$. So where are we in the flow chart? It turns out that this space is pure of dimension 0, so we are in (i). So this is a counting problem: counting covers of elliptic curves. We can write down a generating series

$$\sum_d \langle 1\rangle^E_{1,0,d}q^d = q+\frac{3}{2}q^2+\cdots$$

it turns out its a little bit better to consider the exponential of the above formula:

$$\begin{aligned}Z &= \exp(\sum_d\langle 1\rangle^E_{1,0,d}\, q^d) = 1+q+(\frac{3}{2}+\frac{1}{2})q^2+\cdots\\ &= \sum_{\mu\ partitions} q^{|\mu|}\end{aligned}$$

We can ask: Is there a generalization of the above formula? It turns out that we can and it the subject of some work of Pandharipande-Cooper: The generalization we want to consider is

$$\overline{\mathcal{M}}_1(E \times \mathbb{P}^1, (d_1, d_2)).$$

This space has Vir $\dim_{\mathbb{C}} = 2d_2$. The answer

$$\begin{aligned} Z_1^{E\times\mathbb{P}^1} &= \exp(\sum_{d_1}\sum_{d_2} Q_1^{d_1} Q_2^{d_2} \langle pt, \ldots, pt\rangle^{\text{pure } 1}_{g=1,(d_1,d_2)}) \\ &= \sum_{u,v \text{ partitions}} Q_1^{|u|+|v|} e^{|u|-|v|\sqrt{Q_2}} \end{aligned}$$

Now we move to cohomological field theories. (Kontsevich-Manin)

Definition 0.0.2. *A cohomological field theory (CohFT) consists of the data of a finite dimensional vector space V, over $\mathbb{C}$. A symmetric, nondegenerate, bilinear form $\eta : V \otimes V \longrightarrow \mathbb{C}$ (sometimes called the metric and often written as $\langle\ \rangle \in \mathbb{C}$). Finally we also ask for a distinguished element of V we denote by 1 called the unit. We call $(V, \eta, 1)$ state space. We define the* CohFT *on top of the state space, that is it is a system of tensors*

$$\Omega = \{\Omega_{g,n}\}_{2g-2+n>0}$$

where

$$\begin{aligned} \Omega_{g,n} &\in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \otimes (V^{\otimes n})^* \\ &= \mathrm{Hom}(V^{\otimes n}, H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C})) \end{aligned}$$

which satisfy certain axioms:

(I) $\Omega_{g,n}$ is Σ_n-invariant: The moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ has an action of Σ_n by permuting the markings which induces an action on the cohomology $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$. We also have a Σ_n action on $V^{\otimes n}$ by permutation. We ask that if we have an action of Σ_n on the $\Omega_{g,n}$ it should respect these two actions.

(II) Splitting axiom: First we need to discuss boundary maps: There are boundary maps of two flavors. Firstly, there's

$$q : \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

which glues two special points to create a node. Secondly, there's

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

where $g_1 + g_2 = g$ and $n_1 + n_2 = n$, which is given by connecting curves. We will just describe what happens for q (the one for r is the same). The axiom is that

$$q^*\Omega_{g,n}(v_1, \ldots, v_n) = \sum_{i,j} \Omega_{g-1,n+2}(v_1, \ldots, v_n.e_i, e_j)\eta^{i,j}$$

where we take a basis $e_1, \ldots, e_n$ of V then $\langle e_i, e_j\rangle = \eta_{i,j}$ then write $\eta^{i,j}$ for the inverse matrix.

(III) Forgetting the tails: If you have an extra point, you can forget it, that is you have a map

$$\begin{array}{c}\overline{\mathcal{M}}_{g,n+1}\\ \Big\downarrow p\\ \overline{\mathcal{M}}_{g,n}\end{array}$$

then we ask that

$$\Omega_{g,n+1}(v_1,\ldots,v_n,1) = p^*\Omega_{g,n}(v_1,\ldots,v_n).$$

(There is a degenerate part to this axiom, namely $\Omega_{0,3}(v_1,v_2,1) = \langle v_1, v_2\rangle$*.)*

Gromov-Witten furnishes us with an example of a CohFT: $V := H^*(X,\mathbb{C})$, $\eta :=$ Poincare, $1 = 1 \in H^*(X,\mathbb{C})$,

$$\Omega_{g,n}(\gamma_1,\ldots,\gamma_n) = \sum_\beta j_{\overline{\mathcal{M}}_{g,n}*}(\prod ev_i^*(\gamma_i) \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}})q^\beta \in H^*(\overline{\mathcal{M}}_{g,n})$$

where $\gamma_i \in H^*(X,\mathbb{C})$.

Remark 0.0.3. *Actually if you want to do this correctly you should work in the "super" context. But we wont do this.*

This was all kind of old stuff. Whats new? This past year:

$K3 \times E \rightsquigarrow$ Igusa cusp form: Oberdieck - Pixton, Oberdeick-Shen

Quintic $\rightsquigarrow$ Janda-Ruan, Q. Chen, S. Guo; Holomorphic anomaly equation for quintic.

quintic——————————formal quintic

Givental-Teleman classification