STABILITY CONDITIONS AND WALL-CROSSING IN DERIVED CATEGORIES I

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These talks will be a kind of introduction to space of stability conditions on derived categories and applications to Donaldson-Thomas invariants. Let's recall Donaldson-Thomas Invariants:

$$\text{DT invariants} = \text{vir } \# \text{ of } \underline{\text{(semi) stable sheaves}} \text{ on CY 3-folds}$$

Let's recall the classical definition of stability conditions on coherent sheaves. There several versions of it, here is the definition of so called slope stability conditions:

Definition 0.0.1. *Let X be a smooth projective variety over $\mathbb{C}$, and ω ample divisor on X. A sheaf $E \in \mathrm{Coh}(X)$ is called μ_ω - (semi) stable if: i) E is torsion free, ii) for all $0 \neq F \subsetneq E$ with $rk(F) < rk(E)$, we have $\mu_\omega \underset{(\leq)}{<} \mu_\omega(E)$. Where μ_ω is defined as*

$$\mu_\omega(C) := \frac{c_1(E) \cdot \omega^{\dim X - 1}}{rk(E)} \in \mathbb{Q} \cup \{\infty\},$$

where the value ∞ occurs when $rk(E) = 0$.

In the one dimensional case the definition of μ_ω is independent of the choice of ω but in the higher dimensional cases it will depend on the choice.

<u>Some good things that happen by introducing stability conditions:</u>

i) Let $v \in H^{2k}(X, \mathbb{Q})$ we have

$$M(v) \overset{\text{open substack}}{\supset} M^{ss}_\omega(v) \overset{\text{open}}{\supset} M^s_\omega(v).$$

Where: $M(v)$ is the moduli stack of of coherent sheaves, E, with $ch(E) = v$. It is not of finite type and not separated. $M^{ss}_\omega(v)$ is the open moduli sub-stack of μ_ω-semi-stable sheaves which is of finite type but still not separated. Finally $M^s_\omega(v)$ is the open sub stack of μ_ω-stable sheaves and if you ignore the $\mathbb{C}^*$ automorphisms then it is a quasi-projective variety.

<u>Upshot:</u> If you consider (semi)-stable sheaves you get a better behaved moduli space.

ii) (Harder-Narasimhan) For all $E \in \mathrm{Coh}(X)$ there exists a unique filtration

$$E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$$

such that E_0 is 0 or torsion free, $F_i := E_i/E_{i-1}$ is μ_ω- semi-stable for all i, and

$$\mu_\omega(F_1) > \cdots > \mu_\omega(F_n).$$

Remark 0.0.2. *There is another notion of stability conditions, the so called "Gieseker stability conditions" which involve using the higher chern characters.*

When $M_\omega^{ss}(v) = M_\omega^s$ then you get a projective scheme. Further, if X is a Calabi-Yau 3-fold then

$$DT_\omega(v) := \int_{[M_\omega^s(\nu)]^{\mathrm{vir}}} 1 = \int_{[M_\omega^s(v)]} \nu \cdot de \in \mathbb{Z}$$

where ν-is the Behrend function.

Remark 0.0.3. *There is a generalization of this when $M_\omega^{ss}(v) \supsetneq M_\omega^s(v)$, which is due to Joyce-Song and Kontsevich-Soibelman: $DT_\omega(v) \in \mathbb{Q}$)*

Example 0.0.4. *(MNOP) Let $\beta \in H_2(X,\mathbb{Z})$ and $n \in \mathbb{Z}$ then we can define $I_{n,\beta} := DT_\omega(1,0,-\beta,-n)$ where $(1,0,-\beta,-n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6$. Then*

$$I_{n,\beta} = \{vir\ \#\ of\ C \hookrightarrow X,\ \dim C \leq 1, [C] = \beta, \chi(\mathcal{O}_C) = n\}.$$

You can identify $C \hookrightarrow X$ with a stable sheaf by identifying it with its ideal sheaf. Some of the properties of this invariant are:

i) $I_{n,\beta}$ is independent of ω.
ii) stability $\iff$ torsion free.

Goal: Extend Donldson-Thomas theory to derived categories of coherent sheaves, $D^b(X)$, i.e. want to count stable objects in $D^b(X)$.

Expected applications:

i) If $D^b(X) \cong D^b(Y)$ (e.g. X birational to Y) $\implies$ compare DT invariants on X and Y.

ii) If $\varphi \in Aut(D^b(X))$ $\implies$ get constraints on DT invariants induced by φ.

We now recall the notion of Bridgeland stability. Let D be a triangulated category $(e.g. D = D^b(X))$.

Definition 0.0.5. *A heart of a bounded t-structure on D is a subcategory $A \subset D$ such that:*

i) For all $i < 0$, $\mathrm{Hom}(A, A[i]) = 0$.

ii) For all $E \in D$ there exists

$$\begin{array}{ccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow \cdots & \cdots E_{n-1} & \longrightarrow & E_n = E \\ & \overset{[1]}{\nwarrow} & \downarrow & & & \overset{[1]}{\nwarrow} & \downarrow \\ & & F_1 & \cdots & \cdots & & F_n \end{array}$$

distinguished triangles such that $F_i \in A[k_i]$ where $k_1 > k_2 \cdots > k_n$.

Remark 0.0.6. *If D is a triangulated category with t-structure then the heart of this t-structure is an abelian category.*

Example 0.0.7. *If we take $D = D^b(X)$ then the heart with resepect to the standard t-structure is $A = \mathrm{Coh}(X)E$*

Example 0.0.8. *(Tilting) Let $T, F \subset \mathrm{Coh}(X)$ sub-cats such that*

i) $\mathrm{Hom}(T, F) = 0$

ii) For all $E \in \mathrm{Coh}(X)$ fits into exact sequence

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

where $E_1 \in T$ and $E_2 \in F$.

Such a pair is called a torsion pair. (e.g. If $D = D^b(X)$, you could take T to be the category of torsion sheaves and F to be the category of torsion free sheaves and this would be an example of such a pair). The category

$$\mathcal{A} = \{E \in D^b(X) | \mathcal{H}^0(E) \in T, \mathcal{H}^{-1}(E) \in F, \mathcal{H}^i(E) = 0 \text{ for } i \neq 0, 1\},$$

($= \langle F[1], T\rangle$), is the heart of a bounded t-structure. We call this "tilting with respect to (T, F)". In particular its a procedure that breaks $\mathrm{Coh}(X)$ which itself was a heart of a bonded t-structure into two pieces T, F and then creates a new heart of a bounded t-structure $\langle F[1], T\rangle$.

Moral: In general there can exist more than one "heart of a bounded t-structure" of a triangulated category.

Remark 0.0.9. *Different heart gives different "torsion free" objects.*

Example 0.0.10. *Let X be a 3-fold and consider the category $T := \mathrm{Coh}_0(X) \subset \mathrm{Coh}(X)$ of 0-dim sheaves (i.e. sheaves with 0-dimensional support). Let*

$$F := \{E \in \mathrm{Coh}(X) \mid \mathrm{Hom}(T, E) = 0\}$$

This gives a heart $\mathcal{A} = \langle F, T[-1]\rangle$, $[-1]$ of tilting. Then we can consider

$$\mathcal{A}_{tor} := \{E \in \mathcal{A} \mid rk(E) = 0\}$$

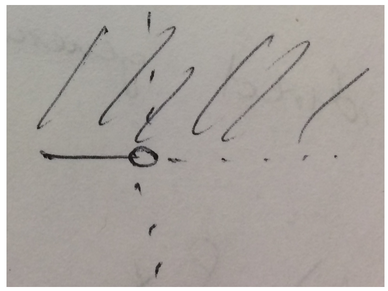
Lemma 0.0.11. *$I \in \mathcal{A}$, $rk(I) = 1$ and $det(I) = \mathcal{O}_X$ satisfies $\mathrm{Hom}(\mathcal{A}_{tor}, I) = 0$ if and only if $I \simeq (\mathcal{O}_X \xrightarrow{s} \mathcal{F})$ such that $\mathcal{F}$ is pure 1-dimensional sheaf and s is surjective in dimension 1. (Pandharipande-Thomas stable pairs)*

Definition 0.0.12. *(Bridgeland Stability Conditions) Let D be a triangulated category. A Bridgeland stability condition on D consists of $\sigma := (Z, \mathcal{A})$ such that:*

i) $Z : K(D) \longrightarrow \mathbb{C}$ *a group homomorphism.*

ii) $\mathcal{A} \subset D$ *heart of bounded* t*-structure.*

(iii) $Z(\mathcal{A} - \{0\})$ *is contained in:*



$\rightsquigarrow E \in \mathcal{E}$, *$\sigma$-(semi)stable if for all* $0 \neq F \subsetneq E$ *in* $\mathcal{A}$ $arg(Z) \underset{(\leq)}{<} arg(E)$ *in* $(0, \pi]$.

(iv) There exists HN filtrations.

Example 0.0.13. *Let* $D = D^b(A - mod)$ *where* A *is a finite dimensional* $\mathbb{C}$*-algebra. Then the standard heart is* $\mathcal{A} = A - mod$*. In this case we have a finite number of simple objects,* $S_1, \ldots, S_N$*, which generate* $\mathcal{A}$*. For the central charge we may take any*

$$Z : K(D) = \bigoplus_{i=1}^{N} \mathbb{Z}[S_i] \longrightarrow \mathbb{C} \tag{0.1}$$

such that $Z(S_i)$ *is contained in*

[picture]

The pair $(Z, \mathcal{A})$ *gives a stability condition.*

Example 0.0.14. *Let* X *be a smooth projective variety,* ω*, ample divisor. Let* $D = D^b(X)$ *and* $\mathcal{A} = \mathrm{Coh}(X)$*. Define*

$$\begin{array}{rcl} Z : K(X) & \longrightarrow & \mathbb{C} \\ E & \mapsto & -c_1(E)\omega^{\dim X - 1} + i \cdot rk(E). \end{array}$$

The pair $(Z, \mathcal{A})$ *is a stability condition if and only if* $\dim X = 1$*. The reason is that in the higher dimensional case we don't capture the higher chern classes in the central charge, indeed we have when* $\dim X \geq 2$ *then* $Z(\mathcal{O}_X) = 0$ *which violates the definition of stability condition.*

The last example shows that Bridgeland stability is not exactly a direct generalization of the classical notion of stability.

Theorem 0.0.15. *(Bridgeland) Fix* $cl : K(D) \longrightarrow \Gamma$ *group homomorphism.*

$$Stab_{\Gamma}(D) := \{(Z, \mathcal{A}) \mid Z : K(D) \xrightarrow{cl} \Gamma \to \mathbb{C}, \text{ support property}\}$$

is a complex manifold such that the forgetful map:

$$\begin{array}{rcl} Stab_\Gamma(D) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) \\ (Z, \mathcal{A}) & \mapsto & Z. \end{array}$$

is a local homeomorphism if $Stab_\Gamma(D) \neq \emptyset$.

Let X be a smooth projective variety and $D = D^b(X)$, and $\Gamma := Im(ch : K(X) \longrightarrow H^{2k}(X, \mathbb{Q}),\ cl := ch$. Then $Stab(X) := Stab_\Gamma(D)$.

Fact: $\nexists (Z, \mathcal{Z}) \in Stab(X)$ such that $\mathcal{A} = \mathrm{Coh}(X)$.

Conjecture: $Stab(X) \neq \emptyset$ (true if $\dim(X) \leq 2$, open in $\dim(X) \geq 2$.)

Application to DT: Let X be a CY 3-fold with $Stab(X)$ non-empty. We expect

$$\begin{array}{rcl} DT_*(v) : Stab(X) & \longrightarrow & \mathbb{Q} \\ \sigma & \mapsto & DT_\sigma(v) \end{array}$$

where $DT_\sigma(v)$ is the virtual number of σ-stable sheaves E, $ch(E) = v$. We also expect $\exists \sigma_x$ (depending on v) with

$$DT_{\sigma_x}(v) = DT_\omega(v).$$

where $DT_\omega(v)$ should count DT.

Suppose now you have an equivalence $D^b(X) \simeq D^b(Y)$, then $Stab(X) \simeq Stab(Y)$ and then wall crossing formula $\implies$ relation of DT invariants of X and Y: