Coulomb branch of a 3d $N=4$ gauge theory

based on N, Braverman–Finkelberg–N, Kodera–N, Takayama–N

Input : · a complex reductive group G
· a symplectic representation $\mathbb{M}$ of G
(need to assume $\mathbb{M} = \mathbb{N} \oplus \mathbb{N}^*$ so far)

Output : $\mathcal{M}_C \equiv \mathcal{M}_C(G, M)$ affine algebraic variety
with ∘ $\mathbb{C}^\times$-action
∘ Poisson structure
s.t. holo. sympl. on regular locus

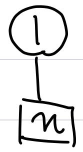
expectations ① $\mathcal{M}_C$ is what physicists call Coulomb branch of 3d $N=4$ SUSY gauge theory
physicists expect hyperKähler metric
our holo. symp. str. should be the underlying one

② closely related to curve counting in Higgs branch $\mathcal{M}_H$
[cf Hikita conjecture] $\qquad$ $\mathcal{M}_H$ = symp. reduction $\mathbb{M} /\!/\!/ G$
e.g. quiver variety $= \mathcal{M}_H$ (some G, $\mathbb{M}$) → Davesh's lectures

③ "Koszul" duality between quantizations of $\mathcal{M}_C$ & $\mathcal{M}_H$
Webster and his collaborators

③′ "Ringel" duality between perverse sheaves on $\mathcal{M}_C$ & $\mathcal{M}_H$
Braden–Mautner, Conjectural geometric Satake for KM alg.

④ applications to 3d/4d gauge theories ?

Ex.  $\quad G = \mathbb{C}^\times$
$\mathbb{M} = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$

$\mathcal{M}_C = \mathbb{C}^2/\mathbb{Z}/_{n-1} \leftarrow \widetilde{\mathbb{C}^2/\mathbb{Z}/_{n-1}}$

$\mathcal{M}_H = \mathcal{N} \leftarrow T^*\mathbb{P}^{n-1}$

flavor of "duality" : same Euler #

Definition

$D = \operatorname{Spec} \underbrace{\mathbb{C}[[z]]}_{=\mathcal{O}}$ formal disk

$D^\times = \operatorname{Spec} \underbrace{\mathbb{C}((z))}_{=K}$ " punctured disk

$$\mathrm{Gr}_G = \text{affine Grassmannian of } G = G_K / G_{\mathcal{O}}$$
$$= \left\{ (P, \varphi) \;\middle|\; \begin{array}{l} P : G\text{-bdle over } D \\ \varphi : P|_{D^\times} \simeq D^\times \times G \text{ trivialization} \end{array} \right\} \Big/ \text{iso.}$$

$$\mathcal{R} = \text{variety of triples}$$
$$= \left\{ (P, \varphi, s) \;\middle|\; \begin{array}{l} P, \varphi : \text{as above} \\ s \in \Gamma(P \times_G \mathbf{N}) \text{ s.t. } \varphi(s) \in \mathbf{N}_{\mathcal{O}} \end{array} \right\} \Big/ \text{isom}$$

$G_{\mathcal{O}} \curvearrowright \mathcal{R}$ $\quad H_*^{G_{\mathcal{O}}}(\mathcal{R})$ $\quad G_{\mathcal{O}}$-equiv. Borel-Moore homology of $\mathcal{R}$

This has a convolution product $*$ preserving the degree.

Heuristic definition

$\mathcal{T} := G_K \times_{G_{\mathcal{O}}} \mathbf{N}_{\mathcal{O}} \longrightarrow \mathbf{N}_K$ $\qquad \mathcal{Z} := \mathcal{T} \times_{\mathbf{N}_K} \mathcal{T} = G_K \times_{G_{\mathcal{O}}} \mathcal{R}$

$(P, \varphi, s) \longmapsto \varphi(s)$

"$H_*^{G_K}(\mathcal{Z})$" $= H_*^{G_{\mathcal{O}}}(\mathcal{R})$

$\mathcal{T} \times_{\mathbf{N}_K} \mathcal{T} \times_{\mathbf{N}_K} \mathcal{T} \xrightarrow{p_{ij}} \mathcal{T} \times_{\mathbf{N}_K} \mathcal{T}$ $\qquad \alpha * \beta = p_{13*}(p_{12}^*(\alpha) \cap p_{23}^*(\beta))$

Rigorous definition $\longrightarrow$ see [BFN]

☆ homological grading is given in relative to $\mathcal{R} \subset \mathcal{T}$

unit = fiber over $(P, \varphi) = (D \times G, \mathrm{id}|_{D^\times})$
degree 0

Thm [BFN]
$(H_*^{G_\mathcal{O}}(\mathcal{R}), *)$ is a $\mathbb{Z}$-graded commutative algebra

Def. $\mathcal{M}_C = \operatorname{Spec}(\text{above})$ affine alg. variety
normal

✩ Poisson str. ⇐ quantization
$\mathbb{C}^\times \curvearrowright D$ loop rotation
$\curvearrowright \mathrm{Gr}_G, \mathcal{R}$ induced action

$(H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}), *)$: noncommutative deformation of $\mathcal{M}_C$
parametrized by $H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

✩ integrable system

$H^*_{G \times \mathbb{C}^\times}(pt) \longrightarrow H_*^{G_\mathcal{O} \times \mathbb{C}^\times}(\mathcal{R}) \rightsquigarrow \mathcal{M}_C \xrightarrow{\varpi} \operatorname{Spec} H^*_G(pt)$
$= \mathfrak{t}/W$

commutative subalg

$\left(\begin{array}{l} \mathfrak{t} = \operatorname{Lie} T \quad T \subset G \text{ max torus} \\ W = \text{Weyl group} \end{array} \right.$

Ex ① $G = \mathbb{C}^\times$, $N = 0$
$\mathcal{R} = \mathrm{Gr}_{\mathbb{C}^\times} \cong \mathbb{Z} \ni m$ $\varphi_m = z^m : \mathcal{O}_D \longrightarrow \mathcal{O}_D$

$\left[\begin{array}{l} \text{half-dim'l} \\ \text{generic fiber} = T^\vee \end{array} \right.$

$H_*^{G_\mathcal{O}}(\mathcal{R}) = \bigoplus_{m \in \mathbb{Z}} \underbrace{H^*_{\mathbb{C}^\times}(pt)}_{= \mathbb{C}[w]}[r^m]$ r^m = fund. class of $\{m\}$

Calculation : $r^m * r^n = r^{m+n}$
$\therefore H_*^{G_\mathcal{O}}(\mathcal{R}) = \mathbb{C}[w, r^\pm]$ $\therefore \mathcal{M}_C = \mathbb{C} \times \mathbb{C}^\times$
symp. form $= dw \wedge \frac{dr}{r}$

In general
We show that
$$\begin{tikzcd} \mathcal{M}_C \arrow[rr, dashed, "\text{birational}"] \arrow[dr] & & \mathfrak{t} \times T^\vee / W \arrow[dl] \\ & \mathfrak{t}/W & \end{tikzcd}$$
$T^\vee$ = dual torus

② $G = \mathbb{C}^\times$, $N = \mathbb{C}$ (wt 1)

$$\mathcal{R} = \coprod_{m\in\mathbb{Z}} \mathcal{O} \cap z^m \mathcal{O} = \coprod_{m\in\mathbb{Z}} z^{\max(m,0)} \mathcal{O} \subset \mathcal{T} = \coprod z^m \mathcal{O}$$

same if $m \geq 0$

In general $H_*^{G_\mathcal{O}}(\mathcal{R}) \longrightarrow H_*^{G_\mathcal{O}}(\mathcal{T}) \cong H_*^{G_\mathcal{O}}(Gr_G)$ is an algebra hom

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ w,\ r^m \ (m\geq 0) & & w,\ r^m \\ w^{-m} r^m \ (m<0) & & \end{array}$$

$x = r$
$y = w r^{-1}$

$\therefore\ xy = w$ $\quad \therefore\ H_*^{G_\mathcal{O}}(\mathcal{R}) = \mathbb{C}[x,y,w]/xy=w$

$$\therefore\ \mathcal{M}_C = \mathbb{C}^2 \xrightarrow{w} \mathbb{C}$$
$$(x,y) \longmapsto w = xy$$

②' $G = \mathbb{C}^\times$, $N = \mathbb{C}$ wt k or $\mathbb{C}^k$ wt 1 $\Rightarrow \mathcal{M}_C = \mathbb{C}^2 / \mathbb{Z}/k$

⊛ Kähler & mass parameters

$\pi_0(\mathcal{R}) = \pi_0(\mathrm{Gr}_G) = \pi_1(G)$

$\therefore\ H_*^{G_\mathcal{O}}(\mathcal{R})$ is $\pi_1(G)$-graded

$\therefore\ \mathcal{M}_C \curvearrowleft \pi_1(G)^\wedge = \mathrm{Hom}(\pi_1(G), \mathbb{C}^\times)$ Pontryagin dual

So $\chi : G \to \mathbb{C}^\times$ Kähler parameter

$\mathcal{M}_H^\chi = \mathbb{M} /\!/\!/_\chi G$

$\rightsquigarrow \pi_1(\chi)^\wedge : \mathbb{C}^\times = \pi_1(\mathbb{C}^\times)^\wedge \longrightarrow \pi_1(G)^\wedge$ 1PS group

Suppose $G \triangleleft \tilde{G} \curvearrowright \mathbb{N}$ s.t. $\tilde{G}/G = T_F$ flavor symmetry
normal subgrp

$\rightsquigarrow T_F \curvearrowright \mathcal{M}_H = \mathbb{M} /\!/\!/_\chi G$

$\tilde{\mathcal{M}}_C := H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}_{\tilde{G},\mathbb{N}}) \curvearrowleft \pi_1(\tilde{G})^\wedge \leftarrow \pi_1(T_F)^\wedge = T_F^\vee$ dual torus

<u>Prop</u> $\mathcal{M}_C = \tilde{\mathcal{M}}_C /\!/\!/ T_F^\vee$

$\rho : \mathbb{C}^\times \to T_F$ 1PS mass

$\leftrightarrow \rho : T_F^\vee \to \mathbb{C}^\times$ Kähler

application : toric hyper-Kähler mfds

$$1 \to T \to \tilde{T} \to T_F \to 1$$
$$\tilde{T} = (\mathbb{C}^\times)^n \curvearrowright_{\text{std}} \mathbb{N} = \mathbb{C}^n$$

$\mathcal{M}_H = \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!/\!/ T$

$$\Rightarrow \mathcal{M}_C = \underbrace{\tilde{\mathcal{M}}_C}_{\overset{\text{Ex②}}{=}\ \mathbb{C}^n \oplus (\mathbb{C}^n)^*} /\!/\!/ T_F^\vee = \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!/\!/ T_F^\vee$$

This is the easiest example.
$\Rightarrow$ used for tests
many conjectures

☆ monopole formula (Cremonesi - Hanany - Zaffaroni)

$G_\mathcal{O} \curvearrowright Gr_G = G_\mathcal{K}/G_\mathcal{O}$ left mult. or $(P,\varphi) \mapsto (P, g\varphi)$, $P|_{D^\times} \xrightarrow{\ \sim\ } D^\times \times G$

Fact. • $G_\mathcal{O}$-orbits are $\{ G_\mathcal{O} z^\lambda \mid \lambda : \mathbb{C}^\times \to T$ dominant cochar $\}$

• closure relation $\Leftrightarrow$ dominance order on cocharacters

e.g. $G = GL_r \Rightarrow (P,\varphi) \simeq$ direct sum of line b'dles ($G_\mathcal{O}$-conj.)

$\mathcal{R} \longrightarrow Gr_G$

$[(P,\varphi,s)] \mapsto [(P,\varphi)]$

fiber is a vector space finite codim. in $\mathbf{N}_\mathcal{O}$

$G_\mathcal{O} z^\lambda$ = vector b'dle over G/P_λ (parabolic subgrp of λ)

Meyer-Vietoris splits for $\mathcal{R}_{<\lambda} \hookrightarrow \mathcal{R}_{\leq\lambda} \hookleftarrow \mathcal{R}_\lambda$

Th. $\mathrm{ch}_{\mathbb{C}^\times} \mathbb{C}[\mathcal{M}_C] = P_t(\mathcal{R}) = \sum_{\lambda:\text{dom. cochar}} t^{2☆(\lambda)} P_t(B\mathrm{Stab}_G \lambda)$

where $☆(\lambda) = \sum_\chi \max(-\langle \chi,\lambda\rangle, 0) \dim \mathbf{N}(\chi) - \sum_{\alpha\in\Delta^+} |\langle \alpha,\lambda\rangle|$ ← wt space

Rk ① RHS is a purely combinatorial expression

② RHS is ∞-sum $\Rightarrow$ may not make sense in general

e.g. $\mathcal{M}_C = \mathbb{C} \times \mathbb{C}^\times \circlearrowleft \mathbb{C}^\times$

Nevertheless $\mathcal{M}_C$ is always well-defined.

★ relation to motivic DT invariants

Σ : proj. curve choose $K_\Sigma^{1/2}$

$G \curvearrowright \mathbb{M}$ $\mu : \mathbb{M} \to \mathfrak{g}^*$ moment map

$\mathcal{M}$ = moduli stack of (P, s)

P : G-b'dle over Σ

$s \in H^0((P \times_G \mathbb{M}) \otimes K_\Sigma^{1/2})$ s.t. $\mu(s) = 0$

Th [Diaconescu]

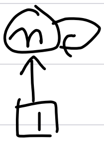
$\mathcal{M}$ carries a canonical symmetric perfect obstruction theory

∵ critical loci of $(A, s) \mapsto \int_\Sigma \omega(\bar\partial_A s \wedge s)$

Diaconescu considered :

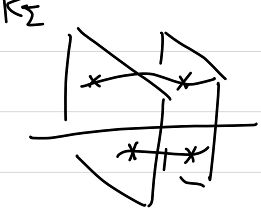
$G = GL(n)$

$\mathbb{N} = \mathfrak{gl}(n) \oplus \mathbb{C}^n$



$\mathcal{M}_H = \mathrm{Sym}^n \mathbb{C}^2 \leftarrow \mathrm{Hilb}^n \mathbb{C}^2$

He showed that (motivic) inv. with suitable stab. cond. = (motivic) usual PT inv. of noncpt CY3 $\mathrm{Tot}(K_\Sigma^{1/2} \oplus K_\Sigma^{1/2})$



Suppose $\mathbb{M} = \mathbb{N} \oplus \mathbb{N}^*$

$\mathcal{M} = \{ (P, s) \mid \underset{s_1 \oplus s_2}{\underset{\shortparallel}{\mu(s)}} = 0 \}$

Cutting ....., Davison $\Rightarrow$ refined PT no stability = motivic inv. of $\mathcal{M}^{red} = \{ (P, s_1) \mid s_1 \in H^0((P \times_G \mathbb{N}) \otimes K_\Sigma^{1/2})$ $(s_2 = 0)$ $\}$

Th. mot. inv. of $\mathcal{M}^{red}$ for $\Sigma = \mathbb{P}^1$ $= \mathrm{ch}_{\mathbb{C}^*} \mathbb{C}[\mathcal{M}_C]$

- Any G-b'dle over $\mathbb{P}^1$ can be reduced to a T-b'dle.

$\Rightarrow$ Same MV argument as before.

each term of $\sum_\lambda$ is equal. //