STABILITY CONDITIONS AND WALL CROSSING IN DERIVED CATEGORIES II

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(*) Notes taken by Dhyan Aranha, all errors should be attributed to me and my ignorance about the subject. Corrections and suggestions are welcome, and should be sent to: dhyan.aranha@gmail.com.

Let X be a smooth projective variety over $\mathbb{C}$. We have a complex manifold $Stab(X)$. Conjecture: $Stab(X) \neq \emptyset$.

Applications to DT theory (2 approaches):

i) Apply wall-crossing for "degenerate" stability conditions

ii) Apply wall-crossing for Calabi-Yau 3-folds with $Stab(X) \neq \emptyset$.

Let's say a little bit about degenerate stability conditions. Take $B + i\omega \in H^2(X, \mathbb{C})$ with ω ample. Define

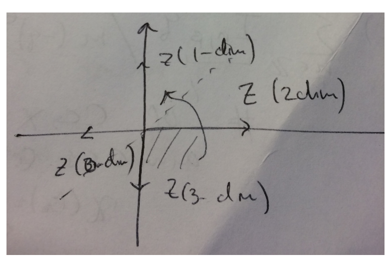
$$\begin{aligned} Z_{B,\omega} : K(X) &\longrightarrow \mathbb{C} \\ E &\mapsto -\int_X e^{-i\omega} ch^B(E) \end{aligned}$$

Expect: $\exists$ a certain heart of t-structure $\mathscr{A}_{B,\omega} \subset D^b(X)$ such that $(Z_{B,\omega}, \mathscr{A}_{B,\omega}) \in Stab(X)$.

For example $\dim(X) = 3$, if you expand the integral and write $v_j^B := \omega^{3-j} ch_j^B$

$$Z_{B,\omega}(E) = -v_3^B + \frac{1}{2}v_1^B + i(v_2^B - \frac{1}{6}v_0^B)$$

If you look at the asymptotic behavior as $\omega \to \infty$ you get a picture:



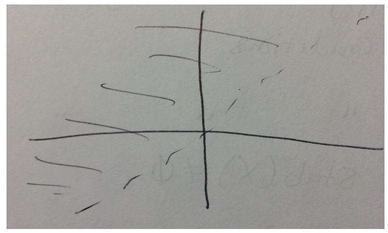
Consider:

$\mathrm{Coh}_{\leq 1}(X) := \{E \in \mathrm{Coh}(X) \mid \dim Supp E \leq 1\}$ $\mathrm{Coh}_{\geq 2} := \{\mathrm{Hom}(\mathrm{Coh}_{\leq 1} X, -) = 0\}$

This is a torsion pair. Then by tilting we get a heart of t-structure

$$^{p}\mathscr{A} = \langle \mathrm{Coh}_{\geq 2}(X)[1], \mathrm{Coh}_{\leq 1}(X)\rangle$$

If $0 \neq E \in^{p} \mathscr{A}$ then the image of the central charge lives in



as $\omega \to \infty$.

Definition 0.0.1. *$E \in^{p} \mathscr{A}$ is $Z_{B,m\omega}$ semi-stable if and only if for all $0 \neq F \subsetneq E$, $Arg\ Z_{B,m\omega}(F) \underset{(\leq)}{<} Arg\ Z_{B,m\omega}(E)$, for $m >> 0$.*

The important thing is that you cannot take pick the m uniformly. i.e. it depends on choice of E. This is the reason why the heart $^{p}\mathscr{A}$ and central charge described above do not give a stability condition.

Wall crossing in $^{p}\mathscr{A} \implies$ Applications to DT-Theory.

Theorem 0.0.2. *(2008 -)*

(i)(DT-PT correspondence) $\sum_{n\in\mathbb{Z}} \frac{I_{n,\beta}q^n}{\mu(-q)^{e(x)}} = \sum_{n\in\mathbb{Z}} P_{n,\beta}q^n (:= P_\beta(X))$

(ii)$P_\beta(x)$ is the Laurent expansion of a rational function of q invariant under $q \mapsto \frac{1}{q}$.

(iii)Suppose you have a flop of CY 3-folds

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z & \end{array}$$

then

$$\phi_* \frac{\sum_\beta P_\beta(x)t^\beta}{\sum_{f_*\beta=0} P_\beta(x)t^\beta} = \frac{\sum_\beta P_\beta(x)t^\beta}{\sum_{g_*\beta=0} P_\beta(x)t^\beta}.$$

<u>Construction of Stability Conditions:</u> Let X be a smooth projective variety of $\dim = d$.

<u>$d = 1$:</u> $Z_{B,\omega}v_1^B + iv_0^B$, $\mathscr{A}_{B,\omega} = \mathrm{Coh}(X)$.

<u>$d = 2$:</u> $Z_{B,\omega} = -v_2^B + \frac{1}{2}v_0^B + iv_1^B$, $\mathscr{A}_{B,\omega}$ tilting of $\mathrm{Coh}(X)$.

<u>$d = 3$:</u> .... double tilting.

Let's recall how to construct the heart in the $d = 2$ case. In general we can take a torsion pair if you have classical slope stability conditions. Let's write

$$\mu_{B,\omega} = \frac{v_1^B}{v_0^B}$$

this defines a torsion pair on $\mathrm{Coh}(X) = \langle T_{B,\omega}, F_{B,\omega}\rangle$. Where $T_{B,\omega}$ is generated by $\mu_{B,\omega}$-semi-stable sheaves with $\mu_{B,\omega} > 0$. $F_{B,\omega}$ is defined in a similar way but with non-positive slope.

By tilting we get

$$\mathscr{B}_{B,\omega} = \langle F_{B,\omega}[1], T_{B,\omega}\rangle.$$

Then we get that $(Z_{B,\omega}, B_{B,\omega}) \in Stab(X)$. This construction first appeared in Bridgeland's work in the case of a $K3$ surface, and Arcara-Bertram. In order that this is a stability condition you need the so called Bogomolov-Gieseker (BG), inequality: for all $\mu_{B,\omega}$ semi-stable sheaves E,

$$(v_1^B)^2 - 2v_0^B v_2^B \geq 0.$$

In the $d = 3$ case the (BG) $\implies$ for all non-zero $E \in \mathscr{B}_{B,\omega}$ we have the following situations:

$$\begin{aligned} v_1^B &> 0, \\ v_1^B &= 0, \quad Im\ Z_{B,\omega} > 0 \\ v_1^B &= 0, \quad Im\ Z_{B,\omega} = 0 \\ & \qquad\quad Re\ Z_{B,\omega} < 0. \end{aligned}$$

So we define

$$\nu_{B,\omega} := \frac{Im\ Z_{B,\omega}}{v_1^B} \in \mathbb{R} \cup \{\infty\}.$$

on $\mathscr{B}_{B,\omega}$. So that $\mathscr{B}_{B,\omega} = \langle T'_{B,\omega}, F'_{B,\omega}\rangle$. Where $T'_{B,\omega}$ is generated by $\nu_{B,\omega}$ semi-stable objects such that $\nu_{B,\omega}$ and similarly $F'_{B,\omega}$ but now $\nu_{B,\omega}$ non-positive. Then via tilting we get $\mathscr{A}_{B,\omega} \subset D^b(X)$.

<u>Conjecture:</u> $(Z_{B,\omega}, \mathscr{A}_{B,\omega}) \in Stab(X)$.

<u>BG-Conjecture:</u> (Bayer-Macri-Toda, Bayer-Macri-Stellari) For all $\nu_{B,\omega}$ - semi-stable $E \in \mathscr{B}_{B,\omega}$,

$$(v_1^B)^2 - 2v_0^B v_2^B + 12(V_2^B)^2 - 18 v_1^B v_3^B \geq 0.$$

The BG-conjecture is known when:

- X is a Fano 3-fold $P(X) = 1$. ($P(X)$:= Picard number of X) (Marcri, Schmidt, Li)

- Some toric 3-folds. (Marcri et al)

• Abelian 3-folds, étale quotients. (Maciocia-Piyaratne, BMS)

• $\mathbb{P}^1 \times ell$, $\mathbb{P}^1 \times$ abelian surface, $\mathbb{P}^1 \times \mathbb{P}^1 \times ell$ (Koseki).
There is a counter example (Schmidt): $X \longrightarrow \mathbb{P}^3$ blow up at a point.

If X Fano 3-fold with picard number > 1 there is a modified version of the conjecuter with is known to be true. (Marci et al, Piyaratne).

• DT invariants on abelian 3-folds. (Toda-Piyaratne-Oberdieck *In progress). Let's assume that A is a principally polarized abelian three fold with Picard number 1.

$$NS(A) = \mathbb{Z}[H]$$

with H ample.

$$\Gamma := Im(ch : K(A) \longrightarrow H^{2*}(A, \mathbb{Q})) = \mathbb{Z} \cdot 1 \oplus \mathbb{Z}[H] \oplus \mathbb{Z}[\frac{H^2}{2}] \oplus \mathbb{Z}[\frac{H^3}{6}] \simeq \mathbb{Z}^{\oplus 4}$$

$SL_2(\mathbb{Z}) \curvearrowright D^b(A)$, modulo shift.

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mapsto \quad \otimes \mathcal{O}_A(H)$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mapsto \quad \Phi_p$$

where Φ_p is the Fourier-Mukai transform with kernel Poincare line bundle.
Can ask: Does this give a constraint on DT-invariants on A via this action?

Let $v \in \Gamma$. Then $DT_H(v) = 0$ due to $A \times \hat{A} \curvearrowright \mathcal{M}_H(v)$. So we can consider

$$DT_H^{red}(v) := \int_{[\mathcal{M}_H(v)/A \times \hat{A}]} v \cdot de \in \mathbb{Q},$$

this is due to Gulbrandsen.

$$\mathscr{C} := \{r(p^3, p^2q, pq^2, q^3) \mid (p, q, r) \in \mathbb{Z}^{\oplus 3}, r \neq 0, gcd(p, q) = 1\}$$

This is nothing but

$$= \{ch(E) \mid E \text{ semi-homogeneous sheaves}\}$$

(due to Mukai). Lets also define $\mathcal{O}(r) := \frac{q}{p} \in \mathbb{Q} \cup \{\infty\}$

Theorem 0.0.3. *(i)* $v \neq \gamma_1 + \gamma_2$ *where* $\gamma_i \in \mathscr{C}$ *and* $\mathcal{O}(\gamma_1) \neq \mathcal{O}(\gamma_2) \implies$

$$DT_H^{red}(v) = DT_H^{red}(g_* v)$$

for all $g \in SL_2(\mathbb{Z})$.

(ii) $v = \gamma_1 + \gamma_2$ *where* $\mathcal{O}(\gamma_1) < \mathcal{O}(\gamma_2)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $-\frac{d}{c} \notin [\mathcal{O}(\gamma_1), \mathcal{O}(\gamma_2))$ $\implies$

$$DT_H^{red}(v) = DT_H^{red}(g_* v).$$

(iii) $v = \gamma_1 + \gamma_2$ *where* $\mathcal{O}(\gamma_1) < \mathcal{O}(\gamma_2)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $-\frac{d}{c} \in [\mathcal{O}(\gamma_1), \mathcal{O}(\gamma_2))$
$\Longrightarrow$

$$DT_H^{red}(v) - DT_H^{red}(g_* v) = (-1)^{r_1 r_2 \alpha} r_1 r_2 \alpha^q \Big(\sum_{\substack{k_1 \geq 1 \\ k_1 | r_1}} \frac{1}{k_1^2}\Big)\Big(\sum_{\substack{k_2 \geq 1 \\ k_2 | r_2}} \frac{1}{k_2^2}\Big)$$

where $r_i = r_i(p_i^3, p_i^2 q_i, p_i q_i^2, q_i^3), \alpha = p_1 q_2 - p_2 q_1$.