Schur algebras

Berkeley -18

§1 "Schur's Lemma"

Finite set $R \curvearrowleft G$, finite group.

Write $r \sim s$ if r and s are in the same G-orbit on R

$(r,s) \sim (t,u)$ if (r,s) and (t,u) — " — " — " — R^2.

For a field F, FR the permutation module.
(char $F = p \geq 0$)

Schur's Lemma. $\mathrm{End}_{FG}(FR)$ has basis

$$\{ \xi_{r,s} \mid (r,s) \in R^2/G \}$$

where

$$\xi_{r,s}(t) = \sum_{\substack{u \in R, \\ (u,t) \sim (r,s)}} u \qquad (t \in R)$$

Moreover,

- $\xi_{r,s} \xi_{t,u} = \sum_{(v,w) \in R^2/G} \#\{x \in R \mid (v,x) \sim (r,s), (x,w) \sim (t,u)\} \xi_{v,w}$
- orthogonal idempotent decomposition $1 = \sum_{r \in R/G} \xi_{r,r}$
- anti-involution $\sim : \xi_{r,s} \mapsto \xi_{s,r}$.

§2. Schur algebra

- Fix $n \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}_{\geq 0}$.

$R = R(n,d) = \{ \underline{r} = r_1 \dots r_d \mid r_1, \dots, r_d \in \{1, \dots, n\} \}$

$\Lambda = \Lambda(n,d) = \{ (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \mid \lambda_1 + \dots + \lambda_n = d \}$

$\Lambda^+ = \Lambda^+(n,d) = \{ \lambda \in \Lambda(n,d) \mid \lambda_1 \geq \dots \geq \lambda_n \}$

- Dominance order on Λ: $\lambda \trianglerighteq \mu \Leftrightarrow \sum_{r=1}^{s} \lambda_r \geq \sum_{r=1}^{s} \mu_r$ for all $s = 1, \dots, n$.

- $\lambda \in \Lambda \rightsquigarrow \underline{r}^\lambda = 1^{\lambda_1} \dots n^{\lambda_n} \in R$

- $R \curvearrowleft \Sigma_d$ via $\underline{r} \cdot \sigma = r_{\sigma(1)} \dots r_{\sigma(d)}$ with orbit reps $\{ \underline{r}^\lambda \mid \lambda \in \Lambda \}$.

Def. $S = S(n,d) = S^F(n,d) := \mathrm{End}_{F\Sigma_d}(F R)$.

- $V = F^n$ with std basis $e_1, \dots, e_n$, then $V^{\otimes d}$ has basis $\{ e_{\underline{r}} := e_{r_1} \otimes \dots \otimes e_{r_d} \mid \underline{r} \in R \}$, the symmetric group Σ_d acts on the right on $V^{\otimes d}$ by place permutations in pure tensors and we can identify $F R$ with $V^{\otimes d}$.

- Schur's Lemma: S has basis $\{ \xi_{\underline{r},\underline{s}} \mid (\underline{r},\underline{s}) \in R^2/\Sigma_d \}$ with Schur's product rule, antiinvolution $\tau: \xi_{\underline{r},\underline{s}} \mapsto \xi_{\underline{s},\underline{r}}$, and orthogonal idempotent decomposition $1 = \sum_{\lambda \in \Lambda} \xi_\lambda$, where $\xi_\lambda := \xi_{\underline{r}^\lambda, \underline{r}^\lambda}$.

Remarks: 1) Let $d = n$, $\omega := (\overbrace{1, \ldots, 1}^{d}) \in \Lambda$

Then $\xi_\omega S \xi_\omega \cong F\Sigma_d$, and we have an exact functor $V \longmapsto \xi_\omega V$, $S\text{-mod} \to F\Sigma_d\text{-mod}$ (equivalence $\Leftrightarrow p = 0$ or $p > d$) James' maxim.

[2). Let $n < N$, and $\xi^N_n := \sum_{\substack{\lambda \in \Lambda(N,d) \\ \lambda_{n+1} = \ldots = \lambda_N = 0}} \xi_\lambda$

Then $\xi^N_n S(N,d) \xi^N_n \cong S(n,d)$, and we have an exact functor $V \longmapsto \xi^N_n V$, $S(N,d)\text{-mod} \to S(n,d)\text{-mod}$ (equivalence $\Leftrightarrow d \le n$)]

§3. Dual construction (F infinite). Let $A(n)$ be the algebra of polynomial functions on $GL_n(F)$, i.e. $A(n) = F[c_{r,s}]_{1 \le r,s \le n}$ where $c_{r,s}(g) = g_{r,s}$.

- The degree d polynomial functions $A(n,d)$ have basis $\{ c_{\underline{r},\underline{s}} := c_{r_1,s_1} \ldots c_{r_d,s_d} \mid (\underline{r},\underline{s}) \in R^2/\Sigma_d \}$.
- Product on $GL_n(F) \rightsquigarrow$ coproduct on $A(n)$ with
$$\Delta(c_{\underline{r},\underline{s}}) = \sum_{\underline{t} \in R} c_{\underline{r},\underline{t}} \otimes c_{\underline{t},\underline{s}},$$
in particular, $A(n,d)$ is a subcoalgebra.
- The dual algebra $A(n,d)^* \cong S(n,d)$ with $\{\xi_{\underline{r},\underline{s}}\}$ being the dual basis of $\{c_{\underline{r},\underline{s}}\}$.

In particular, $A(n,d)$ is naturally an S-module (dual regular module) via

$$\xi \cdot c = \sum_k \xi(c_k')c_k \quad \text{if } \Delta(c) = \sum_k c_k \otimes c_k'.$$

Consequence: $S(n,d)$-mod $= \{$degree d polynomial representations of $GL_n(F)\}$

§4 Quasihereditary algebras. A fundamental fact is that S-mod is a highest weight category, or, equivalently, S is a quasihereditary algebra (CPS). In fact, S is even a quasihereditary algebra with a standard antiinvolution. I will give a slightly unusual definition of what this means (equivalent to the standard one)

Def. Let A be a f.d. algebra with antiinvolution τ, and I be a poset. Then A is based quasihereditary (wrt I) if there are subsets $X(i) \subseteq A$ $(i \in I)$ s.t.

(a) $\bigcup_{i\in I} \{x\tau(x') \mid x, x' \in X(i)\}$ is a basis of A

Set $A^{>i} := \mathrm{span}(x\tau(x') \mid x, x' \in X(j) \text{ for } j > i)$.

(b) $\forall i \in I$, $x \in X(i)$, $a \in A$,

$$ax \equiv \sum_{x' \in X(i)} \overset{F}{\overset{\downarrow}{l_{x,x'}(a)}}\, x' \quad (\mathrm{mod}\ A^{>i})$$

(c) $\forall i \in I$ $\exists$ $e_i \in X(i)$ s.t. $\tau(e_i) = e_i$ and, $\forall x \in X(i), j \in I$:

$$e_i x = \delta_{x,e_i} x$$

$$x e_i = x$$

$$e_j x = x \text{ or } 0$$

$v_x \in A/A^{>i}$

<u>Remarks:</u> 0). $A^{>i}$ is an ideal and $\Delta(i) = \text{span}(x + A^{>i} \mid x \in X(i))$ is a left module with basis $\{v_x \mid x \in X(i)\}$ and the action $a \cdot v_x = \sum_{x' \in X(i)} l_{xx'}(a) v_{x'}$. <u>Standard modules</u> have the following properties:

1). $\text{End}_A(\Delta(i)) = F$

2) $\text{Hom}_A(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i \leq j$

3) Each $\Delta(i)$ has simple head, $L(i)$, and $\{L(i) \mid i \in I\}$ is a complete irredundant set of irreducible A-modules.

4). $\forall i$, $\exists$ a projective A-module P and a surjection $P \to \Delta(i)$ whose kernel has filtration with subfactors of the form $\Delta(j)$, $j > i$.

Properties 1)-4) mean that A-mod is a h/w category

§5. S is quasihereditary

A λ-tableau is a function $[\lambda] \to \{1, \dots, n\}$ (thought of as filling boxes with numbers). A tableau is
row std if its entries increase weakly along the rows,
col std — " — " — " strictly — " — columns ↓
std if row std + col std.

Given any tableau T, define $\underline{r}^T \in R$:= the word obtained by reading the entries of T along the rows as reading a Book.

Theorem (Green '93) S is a based quasihereditary algebra with antiinvolution τ wrt Λ^+ with

$$X(\lambda) = \{\xi_{\underline{r}^T, \underline{r}^\lambda} \mid T \in \mathrm{Std}(\lambda)\} \qquad (\lambda \in \Lambda^+)$$

The main part of Green's Theorem is that

$$\bigcup_{\lambda \in \Lambda^+} \{ \underbrace{\xi_{\underline{r}^T, \underline{r}^\lambda} \xi_{\underline{r}^\lambda, \underline{r}^S}}_{= Y_{S,T} \text{ Std codeterminant}} \mid S, T \in \mathrm{Std}(\lambda) \}$$

is a Basis of S. Green's student Woodcock obtained a straightening algorithm, which implies that standard codeterminants form a spanning set. Then it follows by counting using RSK that they form a Basis.

§6. Standard and costandard modules

- Green's Theorem gives the standard modules $\Delta(\lambda)$ with std. basis $\{v_T \mid T \in \mathrm{Std}(\lambda)\}$ whose formal characters are Schur's symmetric functions ($\Delta(\lambda)$'s are all irreducible if $p=0$ or $p>d$). The action on the basis is explicit but involves Woodcock's straightening.

- The dual picture has been worked out by (Desarmenien, Kung, Doubilet, [illegible]) Rota and collaborators in the 70's and is in some sense a little more natural, because the dual module
$$\nabla(\lambda) := \Delta(\lambda)^{\tau} \cong \{c \in A(n,d) \mid c \cdot g = \lambda(g)c,\ \forall g \in B^-\}$$
where $\lambda : B^- \to F^\times$ is the inflation of the character $\lambda : T \to F^\times$ and $(c \cdot g)(g') = c(g'g)$.

- Rota et al construct an explicit basis $\{w_T \mid T \in \mathrm{Std}(\lambda)\}$ of $\nabla(\lambda)$ as explicit products of determinants ((bi)poly determinants)

- The bases $\{v_T\}$ and $\{w_T\}$ are <u>not</u> dual to each other (Desarmenien mx.)

§7. Linear algebra construction of costandard modules

Explain on example:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & \\ \cline{1-2} \end{array}$$

Basis $\{m_T \mid T \in \mathrm{Colstd}(\lambda)\}$ Basis $\{n_T \mid T \in \mathrm{Rowstd}(\lambda)\}$

$$\begin{tikzcd}
\Lambda^{\lambda'} V \arrow[r, hook] \arrow[rrr, bend left, "\varphi"] \arrow[d, equal] & V^{\otimes d} \arrow[r, "\sigma_\lambda"', "\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} \to \begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}"'] & V^{\otimes d} \arrow[r, two heads] & S^\lambda V \\
\Lambda^2 V \otimes \Lambda^2 V \otimes V \arrow[r, hook] & V^{\otimes 5} & V^{\otimes 5} \arrow[r, two heads] & S^3 V \otimes S^2 V
\end{tikzcd}$$

Basis $\{m_T \mid T \in \mathrm{col\,std}(\lambda)\}$

$$m_{\begin{smallmatrix}1&2&1\\2&3\end{smallmatrix}} = (e_1 \wedge e_2) \otimes (e_2 \wedge e_3) \otimes e_1 \qquad n_{\begin{smallmatrix}1&1&2\\1&3\end{smallmatrix}} = e_1^2 e_2 \otimes e_1 e_3$$

Theorem. (i) $\mathrm{Hom}_{S(\ldots)}(\Lambda^{\lambda'} V, S^\lambda V) = F \cdot \varphi_\lambda$

(ii) $\mathrm{Im}\, \varphi = \nabla(\lambda)$.

(iii) $\varphi(m_T) = w_T$ if $T \in \mathrm{Std}(\lambda)$.

(iv) If $\{k_T \mid T \in \mathrm{Colstd}(\lambda)\}$ is Lusztig's dual canonical basis of $\Lambda^{\lambda'} V$ and $\{\ell_T \mid T \in \mathrm{Std}(\lambda)\}$ is Lusztig's dual canonical basis of $\nabla(\lambda)$, then

$$\varphi(k_T) = \begin{cases} \ell_T & \text{if } T \in \mathrm{Std}(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

§8.4 Categorification

$$V = \widehat{\bigoplus_{r=1}} \mathbb{C} \cdot e_r \rightsquigarrow V = \bigoplus_{r \in \mathbb{Z}} \mathbb{C} \cdot e_r \circlearrowleft \mathfrak{sl}_\infty(\mathbb{C})$$

$$\Lambda^{\lambda'} V \xrightarrow{\ \varphi_\lambda\ } \nabla(\lambda) \hookrightarrow S^\lambda V$$

m_T $\qquad$ w_T $\qquad$ n_T

k_T $\qquad$ l_T $\qquad$ d_T Lower.

translation functors $\circlearrowright$ $M(T)$ parabolic Verma modules; $K(T)$ irreducible modules; $\mathcal{O}^{\lambda'}(\mathfrak{gl}_d)$

integral parabolic category $\mathcal{O}$ over $\mathfrak{gl}_d$ corresponding to parabolic subalgebra of shape λ'

$W(T)$ std.; $L(T)$ irreducible; $\mathrm{tr} \circlearrowright W(\mathfrak{gl}_d, \lambda)$-mod

std. finite dimensional reps over a finite W-algebra correspon-ding to a nilpotent of Jordan shape λ

$W(T)$ Verma's; $J(T)$ irr.; $\mathrm{tr} \circlearrowright \mathcal{O}(W(\mathfrak{gl}_d, \lambda))$

category $\mathcal{O}$ over W

Exact functor $\mathcal{O}^{\lambda'}(\mathfrak{gl}_d) \xrightarrow{\ \Phi\ } W(\mathfrak{gl}_d, \lambda)$-mod

with $\Phi(K(T)) = \begin{cases} L(T) & \text{if } T \text{ is std.} \\ 0 & \text{otherwise.} \end{cases}$

Passing to Grothendieck groups recovers the linear algebra picture.

[Brundan-K '08] Reps of shifted Yangians & finite W-algs

(Losev)

§9 On Schurification.

$$\begin{tikzcd}
\text{q.h. } A \arrow[rr, squiggly, "\text{Schurification}"] & & S^A(n,d), \text{ q-h!} \arrow[d, squiggly, "\text{idempotent cut}"] \\
\text{cellular } \bar{A} \arrow[u, squiggly, "\text{q.h. cover}"] \arrow[rr, dashed] & & S^{\bar{A}}(n,d), \text{ cellular}
\end{tikzcd}$$

- $S^F(n,d)$ is the usual Schur algebra, but can iterate!

Motivation: If $\bar{A} = Z$ a zigzag algebra and $n \geq d$, then $S^Z(n,d)$ describes an arbitrary block of symmetric groups up to derived equivalence. Broué's Conjecture provides such a conjectural description for blocks of finite groups provided the defect group of a block is abelian. For non-abelian defect groups, there is no conjecture and in this sense $S^Z(n,d)$ provide a first glimpse into an arbitrary defect world.

- A based quasihereditary wrt I, X. Importantly in key examples, A could be a superalgebra in which case we insist that the data X respects this structure, in particular, $X = X_{\bar{0}} \sqcup X_{\bar{1}}$.

$'S^A(n,d) := \left(M_n(A)^{\otimes d}\right)^{\Sigma_d}$. This algebra has an obvious analogue of Schur's basis, but it doesn't have to be quasihereditary.

Suppose A is defined over integers (based approach handy!), define an appropriate notion of X-standard tableaux and use it to define an analogue of codeterminants:

$$\left\{ Y^{\underline{\lambda}}_{\underline{S},T} \;\middle|\; \underline{\lambda} \in \Lambda^I_+(n,d) \;\middle|\; \underline{S}, \underline{T} \in \mathrm{Std}^X(\underline{\lambda}) \right\}.$$

Theorem (K-Muth'17) Let $d \leq n$. The

$$S^A(n,d)_{\mathbb{Z}} := \mathrm{span}_{\mathbb{Z}}\left(Y^{\underline{\lambda}}_{\underline{S},\underline{T}}\right) \subseteq {}'S^A(n,d)_{\mathbb{Z}}$$

is a full sublattice, closed under multiplication, and unital. Moreover, it is a based quasihereditary $\mathbb{Z}$-algebra.

- Extending scalars to F gives a based q.h. F-algebra
- New combinatorics, interesting decomposition numbers.