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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rebekah Palmer **Email/Phone:** palmer@temple.edu (443) 621-7023

Speaker's Name: Britta Späth

Talk Title: An overview over the inductive conditions for the global-local conjectures

Date: 01 / Feb / 18 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: representation theory, groups, McKay Conjecture, inductive conditions, induction, reduction theorems, simple groups

Please summarize the lecture in 5 or fewer sentences:
We give a short survey on the reduction theorems of global-local conjectures and their verification so far. We focus on the Clifford-theoretic methods involved and the local structures of simple groups considered. In the end, we report on first steps towards the verification of the Alperin-McKay conditions. (In joint work with J. Brough)

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(An overview over) the inductive Conditions for the global/local Conj. 1

§1. ~~A~~ Origin

What is meant by that? Background? matters

representation theory of finite groups: G finite group, p prime

$\mathrm{Irr}(G) \leftrightarrow$ simple $\mathbb{C}G$-modules $\qquad$ simple $\overline{\mathbb{F}}_p G$-modules $\leftrightarrow \mathrm{IBr}(G)$

(Idea behind Broué's abelian defect gp conj)

Global-local principles:

* (Tool in classification of finite simple groups): local 2-structures $\rightsquigarrow$ CFSG
* Brauer correspondence: $\mathrm{Bl}(G|D) \leftrightarrow \mathrm{Bl}(N_G(D)/D)$ p-blocks of $G \to$ p-blocks of certain subgroups $\to$ Radha Kessar
  already present in Klesheler-talk
* Green correspondence: bij. for certain indecomp. modules
* If G group/Lie type: Deline-Lusztig induction, Harish Chandra induction

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Goal: Global/local principle for $\mathrm{Irr}(G)$ or $\mathrm{IBr}(G)$ $\quad$ E. Idea

McKay Conjecture '72: $\qquad |\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N_G(P))|$

for $P \in \mathrm{Syl}_p(G)$ and $\mathrm{Irr}_{p'}(H) := \{\chi \in \mathrm{Irr}(H) \mid p \nmid \chi(1)\}$

true for p-solvable groups (Isaacs, Okuyama-Wajima) ⊛

Red. theorem (Isaacs-Malle-Navarro '07): McKay Conjecture holds for G, if the inductive McKay conditions hold for all simple groups involved in G.

HERE the inductive cond. show up | ⚠ ⊛ What is missing from simple groups?

PATIENCE $\quad$ inductive conditions are technical $\rightsquigarrow$ later

§2. More Conjectures

· Notation

~~Blockwise~~: p-block of $G \rightsquigarrow$ defect group D, $\mathrm{Irr}(B)$, $\mathrm{IBr}(B)$, b

determines

p-subgroup of G

$\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G)$, $\mathrm{IBr}(B) \subseteq \mathrm{IBr}(G)$, b p-block of $N_G(D)$

p-modular system

$$\mathrm{Irr}_0(B) = \{\chi \in \mathrm{Irr}(B) \mid \chi(1)_p |D| = |G|_p\}$$

p-Part of degrees minimal

Alperin-McKay (AM) Conjecture: $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|$

Alperin's weight (AW) Conjecture $|\mathrm{IBr}(B)| = |\mathrm{IBr}(b)|$ if D is abelian

alternative ...

Brauer's height 0 Conjecture: D is abelian $\Leftrightarrow$ $\mathrm{Irr}(B) = \mathrm{Irr}_0(B)$

again all known for p-solvable groups

~~And~~ refinement's by Gabriel - Galois autom $\to$ Carolina, Mandy

they are implied by Broué's Conjecture and Dade's Conj.

(both are more technical; only for D abelian)

Gabriel, lecture series

§3. Reduction theorems

$\frac{AM}{AW}$ Conjecture holds, if the inductive $\frac{AM}{AW}$ conditions are true for all simples

(S., Navarro-Tiep, Puig)

BHZ: "$\Rightarrow$" true (Berger-Knörr ('88), Kessar-Malle ('12))

"$\Leftarrow$" true, if inductive AM conditions hold

(Navarro-S., Kessar-Malle)

analogous result for Dade Conjecture (S.), prereduction BADC (Craven-Rouquier)

§4 The inductive conditions

lin bes

Reduction theorems are proven by induction on $|G/Z(G)|$

Main idea: Control Clifford theory

If $X \lhd G$ | in general:

$Q \leq G/X$

$\hat{Q}$ central extension

$$\begin{tikzcd} & \mathrm{Irr}(G) & \\ \mathrm{Irr}(X) \arrow[ur, squiggly] & & \mathrm{Irr}(\hat{Q}) \arrow[ul, squiggly] \end{tikzcd}$$

G acts on X and $\mathrm{Irr}(X)$

Example: If every $\psi \in \mathrm{Irr}(X)$ extends to some $\tilde{\psi} \in \mathrm{Irr}(G_\psi)$, then

$$\mathrm{Irr}(G) = \{ \mathrm{Ind}_{G_\psi}^{G}(\tilde{\psi}\eta) \mid \psi \in \mathrm{Irr}(X), \eta \in \mathrm{Irr}(G_\psi/X) \}$$

Inductive McKay conditions for a simple S: [B]

Let X be the universal covering group of S and $P \in \mathrm{Syl}_p(\cancel{X})$.

Then $\exists$ $\mathrm{Aut}(X)_P$ - equiv. $\Omega : \mathrm{Irr}_{p'}(X) \xrightarrow{1:1} \mathrm{Irr}_{p'}(N_X(P))$,
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \psi \mapsto \psi'$

s.t. $\forall \psi \in \mathrm{Irr}_{p'}(X)$:

ψ and ψ' have the same Clifford theory in any $X \lhd G$,
$N_X(P) \lhd N_G(P)$

$$\begin{tikzcd}[column sep=small]
 & & G \\
\psi \; X \arrow[urr, dash] \arrow[dr, dash] & & & N_G(P) \arrow[ul, dash] \\
 & \psi' \; N_X(P) \arrow[urr, dash] & &
\end{tikzcd}$$

i.e. $\exists$ a group A: $X \lhd A$ and $A/C_A(X) \cong \mathrm{Aut}(X)_\psi$

- ψ extends to some $\tilde\psi \in \mathrm{Irr}(A)$
- ψ' " " some $\tilde\psi' \in \mathrm{Irr}(N_A(P))$
- $\mathrm{Irr}(\tilde\psi|_{C_A(X)}) = \mathrm{Irr}(\tilde\psi'|_{C_A(X)})$

have analogues with proj rep.

Remark:
- lead to a better control of Clifford theory
- can be adapted to blocks $\to$ ind $\substack{AM \\ AW}$ conditions
- ideas used in other contexts $\quad |\mathrm{IBr}_A(G)| = |\mathrm{IBr}\, C_A(G)|$

Main parts of the proof go from

$X \lhd A$ (ψ) to $X^r/Z \lhd G$ ($\psi \times \dots \times \psi$)

wreath products

construction butterfly Thm

AM: $bl(\tilde\psi_J) = bl(\tilde\psi'_{N_J(P)})$ $\forall X \leq J \leq A$

§5. Verification

* for ind. AM/AW conditions:

$\cancel{\mathfrak{A}}_n$, most sporadic, „def. characteristic case, low ranks
(S, Malle) (An-Dietrich, Breuer) (S.) ; ..g, Feng, Li (x) Schaeffer-Fry, Schulte, Zhang

~~nilpotent blocks~~, cyclic defect (Ko-Sp), ~~some~~ blocks type A with max defect

* inductive McKay conditions:

- $\mathfrak{A}_n$ $(n \geq 5)$, sporadics, (Malle)
- groups of Lie type ~~(p|q)~~: bijection (Malle, S) (if $p \nmid q$)

  Jordan decomp. ~~of~~ HC Param $\mathrm{Irr}(N_G(P))$

  ind cond. ok, if $p \mid q$ (S.)

  in 10 of 16 types (Cabanes-S.)

<u>Thm (Malle-S.)</u> $|\mathrm{Irr}_{2'}(G)| = |\mathrm{Irr}_{2'}(N_G(P))|$ for any G

§6. and next?

if there is ever a natural bijection it should have automatically the Clifford theoretic properties present in inductive conditions