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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rebekah Palmer **Email/Phone:** palmer@temple.edu (443) 621-7023

Speaker's Name: Carolina Vallejo

Talk Title: Sylow normalizers and Galois action on characters

Date: 01 / Feb / 18 **Time:** 02 : 15 **am / pm (circle one)**

List 6-12 key words for the talk: representation theory, groups, Navarro Conjecture, McKay Conjecture, group actions, Sylow, blocks, normalizers

Please summarize the lecture in 5 or fewer sentences: The Navarro conjecture states that the actions of a particular subgroup of Galois automorphisms on the two sets of characters involved in the McKay conjecture should be permutation isomorphic. Recently verified for all primes, this predicted that the local condition that a Sylow p-subgroup of a finite group is self-normalizing can be characterized in terms of the character theory of the group. We focus on the character theory of the principal p-block and its relation with the structure of the normalizer.

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Sylow normalizers and Galois action on characters

1. The Navarro conjecture (aka Galois McKay conj.)

G finite gp, $|G| = n$, $p \mid n$ prime, $P \in Syl_p(G)$

McKay conj $\quad \# \mathrm{Irr}_{p'} G = \# \mathrm{Irr}_{p'} N_G(P)$,

$\mathrm{Irr}_{p'} H = \{ \psi \in \mathrm{Irr} H \mid p \nmid \psi(1) \}$

Ex if H is a p-gp, then $\mathrm{Irr}_{p'} H = \mathrm{Lin}\, H = \mathrm{Hom}(H, \mathbb{C}^\times) \cong H/H'$

1972, McKay observed $\# \mathrm{Irr}_{2'} G = 2^a$ ~~$\forall G$ (simple)~~ for many G simple

conjectured $\# \mathrm{Irr}_{2'} G = \# \mathrm{Irr}_{2'} N_G(P) \quad \forall G$ simple

1973, Isaacs proved the McKay conj

G solvable $\quad p = 2$
$|G|$ odd $\quad$ all p $\Big\}$ by explicitly constr. the bijections

2016, Malle and Späth proved McKay conj holds for $p=2$ for every gps (CFSG).

$\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q})$ acts on $\mathrm{Irr} H \quad \forall H \leq G$

$\psi^\sigma(h) = \psi(h)^\sigma \leftarrow$ sums of roots of unity

$\psi^\sigma(1) = \psi(1)$

$$\mathrm{Irr}_{p'} G \overset{1:1}{\longleftrightarrow} \mathrm{Irr}_{p'} N_G(P)$$
$\circlearrowleft \mathcal{G}$ $\qquad\qquad \circlearrowleft \mathcal{G}$

are permutation isom?

$\Updownarrow$

$\mathcal{A} \leq \mathcal{G} \quad \# \mathcal{A}$-fixed ch. is preserved

$\Longrightarrow \#$ rational characters is preserved. Too much)

Ex $GL(2,3)$ $\quad p = 3 \quad D_{12}$ rational gp

$\chi(1) = 2 \quad \chi(g) = \pm(\omega + \omega^3) \quad o(\omega) = 3$

However, Isaacs' bijections are $\mathcal{G}$-equivariant.

Thm (Navarro, 04). G p-solvable $\quad N_G(P) = P$

$\exists\ \mathcal{G}$-equiv. bij $\mathrm{Irr}_{p'} G \longrightarrow \mathrm{Lin}\, P$

$\mathcal{H} \le \mathcal{G}$ $\quad \mathcal{H} \cong Gal(\mathbb{Q}_p(\xi_n)/\mathbb{Q}_p)$

$\{\sigma \in \mathcal{G} \mid \exists f \ge 0 \quad \sigma(\xi) = \xi^{p^f} \quad \forall p \nmid o(\xi) \in \mathbb{Q}(\xi_n)\}$

Conjecture (Navarro, 04).

$\mathrm{Irr}_{p'} G \circlearrowleft \mathcal{H}$ $\quad \mathrm{Irr}_{p'} N_G(P) \circlearrowleft$ are permutation isomorphic.

Ex $\tau \in \mathcal{H}$ $\quad \tau(\omega) = \omega^{3^f} = \omega, \omega^3$

$o(\omega) = 8$ ✓ $(f=0)$

$\Longrightarrow$ $\underbrace{\#\,\mathcal{K}\text{-fixed characters}}_{\text{III}}$ is the same, $\mathcal{K} = Gal(\mathbb{Q}(\xi_n)/\mathbb{Q}(\xi_n))$

p-rational characters $\quad n = p^a m$, $p \nmid m$ $\quad \mathcal{K} \le \mathcal{H}$

Ex if H is a p-group, then ψ is p-rational iff ψ is rational

Evidence: p-solvable (Turull)

A_n $\quad p = 2$ (Nath) $\quad \sqrt{k}, i$

p odd (Brunat-Nath)

2. Self-normalizing Sylows

$\overrightarrow{NC}$ "$N_G(P) = P$ can be char. in terms of $\mathrm{Irr}_{p'} G \circlearrowleft \mathcal{H}$"

p odd: in terms of p-rat. ch.

$p = 2$: " " " σ-inv. characters

$$\sigma(\xi) = \begin{cases} \xi & o(\xi) = 2^a \\ \xi^2 & 2 \nmid o(\xi) \end{cases} \qquad \sigma(\sqrt{k}) = \pm\sqrt{k} \text{ dep on } k \bmod 8.$$

Thm (Navarro, Tiep, Turull, 2007). p odd

$N_G(P) = P$ iff $\#$ p-rat $\mathrm{Irr}_{p'} G = 1$

Guralnick, Malle, Navarro (CFSG) $\longrightarrow$ G solvable or $p = 3$ and c.fac. $L_2(3^{3^a})$

Thm (Schaeffer-Fry, 2018). $p = 2$

$N_G(P) = P$ iff every $\mathrm{Irr}_{2'} G$ is σ-fixed.

$\searrow$ wild case! comp. factors are out of control

3 Sylow normalizers with a normal p-complement

$N_G(P) = P \rtimes V$ ~~~ $\mathrm{Irr}_{p'} B_0 \subseteq \mathrm{Irr}_{p'} G$ $\circlearrowleft \mathcal{H}$

"motivation"

- extending Navarro's $\mathcal{G}$-equiv. bij for ~~p-solvable~~ $N_G(P) = P$ p odd
- Navarro's conjecture (Galois-McKay) admits a blockwise version (Galois - Alperin - McKay).

Blocks (briefly) $\bar{k} = k$ char $k = p$ $\circlearrowright \mathcal{G}$

$kG = \underbrace{B_0 \oplus \cdots \oplus B_\ell}_{\mathrm{Bl}(G)}$ - - - → $\mathrm{Irr}\, G = \bigcup_{B \in \mathrm{Bl}(G)} \mathrm{Irr}\, B$

$B \in \mathrm{Bl}(G) \to D \leq G$ p-group ← rather complicated way
defect gp of B

$B \in \mathrm{Bl}(G \mid D)$
$\downarrow$ 1:1 Brauer's corresp. $(b^G = B)$ - - - → $\mathrm{Irr}_{p'} G = \bigcup_{B \in \mathrm{Bl}(G \mid P)} \mathrm{Irr}\, B$
$b \in \mathrm{Bl}(N_G(D) \mid D)$ $\downarrow$ 1:1 (AMC)

$\mathrm{Irr}_{p'} N_G(P) = \bigcup_{b \in \mathrm{Bl}(G \mid P)} \mathrm{Irr}\, b$

$b^G = B$

$B \in \mathrm{Bl}(G \mid P)$ iff $\underbrace{\mathrm{Irr}_{p'} G \cap \mathrm{Irr}\, B}_{= \mathrm{Irr}_{p'} B} \neq \emptyset$

Def $B_0 = B_0(G)$ principal (p-)block of G

$1_G \in \mathrm{Irr}\, B_0 \to B_0 \in \mathrm{Bl}(G \mid P)$

Say $b_0 = B_0(N_G(P))$ Then $b_0^G = B_0$ (Brauer's third main thm) Note $\mathcal{G}$ acts on $\mathrm{Irr}\, B_0$ as $1_G^\sigma = 1_G$ $\forall \sigma \in \mathcal{G}$

Conjecture (Navarro, 2004)

$\mathrm{Irr}_{p'} B_0$ $\circlearrowleft \mathcal{H}$ and $\mathrm{Irr}_{p'} b_0$ $\circlearrowleft \mathcal{H}$ are permutation isom.

Theorem A (NTV, 2014) p odd, $N_G(P)=P\times V$ (CFSG)

$\exists\ \mathcal{G}$-equiv $\mathrm{Irr}_{p'}B_0 \longrightarrow \mathrm{Irr}_{p'}b_0$ bij

- if $N_G(P)=P$, these $\mathrm{Irr}_{p'}B_0 = \mathrm{Irr}_{p'}G \longrightarrow \mathrm{Lin}\,P$
  recover Navarro's bijection from 2004.
- $\#\,p$-rat $\mathrm{Irr}_{p'}B_0 = \#\,p$-rat $\mathrm{Irr}_{p'}b_0 = \#\,\mathrm{rat}\,\mathrm{Lin}\,P = 1$
  p odd

Theorem B (NTV, 2018). p odd (CFSG)

$N_G(P)=P\times V$ iff $\#\,p$-rat $\mathrm{Irr}_{p'}B_0 = 1$

(kill V, by $\chi\in\mathrm{Irr}\,B_0$ show $\chi_V=\chi(1)1_V$)

Conjecture C $p=2$

$N_G(P)=P\times V$ iff every $\mathrm{Irr}_{2'}B_0$ is σ-fixed.

Towards a reduction... 'relate $\mathrm{Irr}_{p'}G$ and $\mathrm{Irr}\,V$'

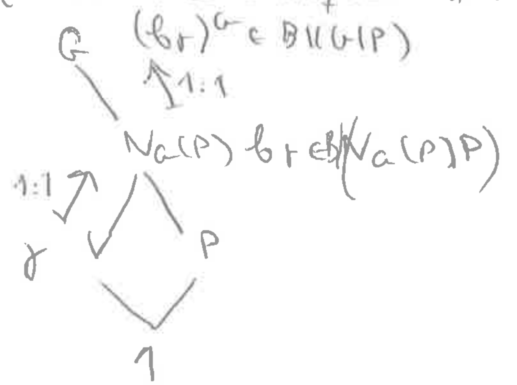
parametrize the blocks of max. defect of G

Theorem D (NV, 17). $N_G(P)=P\times V$ (any p) [block-theory! NO CFSG]

Then $\exists\ \mathcal{G}$-equivariant surjection

$$\mathrm{Irr}_{p'}G \longrightarrow \mathrm{Irr}\,V \qquad \text{("highly non-inj.")}$$
$$\chi \longmapsto \hat\chi$$

$\hat\chi$ is the unique $\gamma\in\mathrm{Irr}\,V$ s.t. $\chi\in\mathrm{Irr}(b_\gamma{}^G)$



actually

$$\chi_V = e\hat\chi + p\Delta \qquad p\nmid e.$$

Rmk Conj C implies the block-free version (SF-Thm)

Pf.

- $N_G(P) = P$, $\mathrm{Irr}_{2'}G = \mathrm{Irr}_{2'}B_0$ are σ-fixed ✓.

- $\mathrm{Irr}_{2'}G$ are ~~f~~ σ-fixed, then $N_G(P) = P \times V$

$\gamma \in \mathrm{Irr}\,V$ $\xrightarrow{\text{by Thm D}}$ $\chi \in \mathrm{Irr}_{2'}G$ $\quad \hat{\chi} = \gamma$

$\chi^\sigma = \chi \Rightarrow \gamma^\sigma = \gamma$

$\Downarrow$ $|V|$ odd

$V = 1$.

Thm E (NV, 17). Conjecture C holds for every gp, if it holds for every almost simple H with $|H/\mathrm{soc}(H)| = 2^a$.

Wait for Mandi to hear the end of the story...

Comments: ~~why~~ interesting $\mathcal{G}$-equivariant bij. $\Rightarrow$ very strong structural consequences for G

~~and~~ local structure influences a lot char. th. of G and the other way around

extensions of Thm B or Conj C for non-principal blocks

Navarro implies that
p odd $N_G(P) = P \times V$ $\Longrightarrow$ $\nexists$ p-rat $\mathrm{Irr}_{p'}B = 1$ (NOT KNOWN IN GENERAL)

$B \in \mathrm{Bl}(G|P)$

$b^G = B$ $\quad b \in \mathrm{Bl}(N_G(P)|P)$

$(\Longleftarrow)$? would say false.

$l(b) = 1$ $\quad \Longleftarrow$ conj

$\boxed{p \text{ odd}}$

$B \in Bl(G|P)$

$N_G(P) = P \times V \implies \nexists$ p-rat $\mathrm{Irr}_{p'}B = 1$ open

"NC"

$\not\Leftarrow$

$G = \text{SmallGroup}(200, 24)$ $p = 5$

here $N_G(P) = G$ $(P \trianglelefteq G)$

P abelian

$\ell(b) = 1 \xLeftarrow[\text{conj}]{}$

$b^G = B$ Br. corr.

Problem to generalize is "what would be the condition on the right"