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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rebekah Palmer **Email/Phone:** palmer@temple.edu (443) 621-7023

Speaker's Name: Sarah Witherspoon

Talk Title: From groups to Hopf algebras: Cohomology and varieties for modules

Date: 02 / Feb / 18 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: representation theory, groups, homology, cohomology, Hopf algebras, support varieties, quantum groups, projective modules

Please summarize the lecture in 5 or fewer sentences: To a group action on a vector space, one associates a geometric object called its support variety that is defined using group cohomology. Hopf algebras generalize groups and include many important classes of algebras such as Lie algebras and quantum groups. The theory of varieties for modules generalizes to Hopf algebras to some extent. We define Hopf algebras, their cohomology, and the corresponding varieties for modules.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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MSRI Workshop
Connections for Women: Group Representation Theory and Applications
Feb 2, 2018, 11 am – 12 pm

From Groups to Hopf Algebras: Cohomology and Varieties for Modules

— mention that I'm grateful Jon Alperin [illegible] (of Alperin's weight conjecture)

Thank organizers & MSRI for its wonderful hospitality. great to see so many students and postdocs here.

<u>Motivating Question</u> For a given group or ring, (what are all of its reps?)
Can we understand all its representations (or modules) in some meaningful way?
i.e. actions on vector spaces etc

I will talk about one method to organize such info that started with finite groups where there is a very rich theory – passing through group cohomology – that Jon Carlson spoke about this morning. I will talk about generalizations to an important class of rings known as Hopf algebras, because one big application of group representation theory is to other settings that involve groups in some way or to which analogous techniques may be applied.

Hopf Algebras

$\boxed{k = \bar{k}}$

<u>Defn</u> A <u>Hopf</u> algebra is an algebra A (over a field k — or other commutative ring) together with maps
$$\Delta : A \to A \otimes A, \quad \varepsilon : A \to k, \quad S : A \to A$$
satisfying some properties (that mimic properties of groups) ($\otimes$ is $\otimes_k$)

Examples (1) $A = kG$, group algebra, $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $S(g) = g^{-1}$ $\forall g \in G$

(2) $A = \mathcal{U}(\mathfrak{g})$, univ. env. alg. of Lie alg $\mathfrak{g}$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, $S(x) = -x$ $\forall x \in \mathfrak{g}$

e.g. $\mathcal{U}(\mathfrak{sl}_2) = k\langle e, f, h \mid fe = ef - h,\ he = eh + 2e,\ hf = fh - 2f \rangle$

(3) $A = U_q(\mathfrak{g})$, quantum group $(q \in k)$

(4) $A = u_q(\mathfrak{g})$, small quantum group $(q^n = 1)$

have one been & understand [illegible]

e.g. $u_q(\mathfrak{sl}_2) = \mathbb{C}\langle e, f, k \mid e^n = 0,\ f^n = 0,\ k^n = 1,$
$kek^{-1} = q^2 e,\ kfk^{-1} = q^{-2} f,$
$ef - fe = \frac{k - k^{-1}}{q - q^{-1}} \rangle$

$\Delta(e) = e \otimes 1 + k \otimes e, \quad \Delta(f) = f \otimes k^{-1} + 1 \otimes f, \quad \Delta(k) = k \otimes k$

<u>Fact</u> Hopf alg A-modules may be "added" (direct sum) and "multiplied" (tensor product).
(get a "tensor category" or ring)

generalizing [illegible] ring of a finite group

(Hopf algs arose first in topology, also in rep th., math. phys., combinatorics, quantum computing ...)

From now on: A is a (fin.dim.) Hopf algebra, k is an A-mod via ε ("trivial" A-mod)

Hopf algebra cohomology

Notation: $H^*(A,k) := \mathrm{Ext}^*_A(k,k) := \bigoplus_{n\geq 0} \mathrm{Ext}^n_A(k,k)$

where one defn of $\mathrm{Ext}^n_A(k,k)$ is as equivalence classes of "n-extensions" of A-modules:

$$k \longrightarrow M_n \to M_{n-1} \to \cdots \to M_1 \to k$$

Remarks (1) $H^*(A,k)$ has a sum (Baer sum) and product (Yoneda splice)

(2) $H^*(A,k)$ is graded commutative

(3) For an A-mod M, define $H^*(M) := \mathrm{Ext}^*_A(M,M)$ similarly.

Then $H^*(M)$ is an $H^*(A,k)$-module via "$- \otimes M$" (explain) (colored diagram - twist to always n-ext)

Example $A = kG$ $\quad H^*(A)$ is group cohomology $H^*(G,k)$ as in Jon Carlson's talk

Etingof-Ostrik '04 but others had thought of it before that

Conjecture $H^*(A,k)$ is finitely generated. (equiv. Noetherian under these conditions: gr commut., fin. dim.)

Known to be true for:

char p (1) $A = kG$ ((Golod '59, Venkov '59, Evens '61))

(2) (more generally) A cocommutative (explain) (Friedlander-Suslin '97, Friedlander-Parshall '83)

char 0 (3) $A = u_q(\mathfrak{g})$, small quantum group (Ginzburg-Kumar '93, Bendel-Nakano-Parshall-Pillen '14)

(4) (more generally) A "pointed" under some conditions (Mastnak-Pevtsova-Schauenburg-W '10)

(5) some others (and much current research on various types by many groups of mathematicians)

e.g. recent results: Erdmann-Solberg-Wang, Nguyen-Wang-W

Caution: A related conjecture for Hochschild cohomology is false!

(Problem is we don't (yet) know enough about fin.dim. Hopf algs to get a general result)

Varieties for Modules

Big Picture:

fin. dim. Hopf alg A $\rightsquigarrow$ fin. gen. graded commut alg. $H^*(A,k)$ (and a geom. space assoc. to it)

fin. dim. A-module M $\rightsquigarrow$ $H^*(A,k)$ module $\mathrm{Ext}^*_A(k,M)$ (and a geom. space assoc. to it)

From now on: Assume A is such that $H^*(A)$ is fin. gen. and $H^*(M)$ is fin. gen. $H^*(A)$-mod (for all fin. dim. M)

use Jon's notation

Defn (support variety)

$V_A(k) :=$ max ideals of $H^*(A,k)$ (max ideals under Zariski topology)

$V_A(M) :=$ max ideals containing annihilator J of $\mathrm{Ext}^*_A(M,M)$

In all settings in earlier list, varieties for modules have been studied to a greater or lesser extent:

(1) Quillen '71, Avrunin-Scott '82, Carlson '83, ...

(2) Suslin-Friedlander-Bendel '97, Friedlander-Pevtsova '05, '07, ...

(3) Parshall-Wang '93, Ostrik '98, ... (use Balmer spectrum (related) to classify thick tensor ideals for $U_q(\mathfrak{g})$)

(4) not much known - some beginnings (e.g. some work of mine w/ Pevtsova & w/ Feldvoss!

Some applications of varieties for modules ~~are to "determining" representation type,~~ to "determining" structure of module category ((e.g. classifying thick tensor ideals)

Useful in these applications are -

Properties

(1) $V_A(M) = \{0\}$ iff M is projective

((more generally: $\dim(V(M)) =$ complexity of M (rate of growth of minimal proj. res.)))

(2) $V_A(M \oplus N) = V_A(M) \cup V_A(N)$ (also SES statement)

~~(3) (realization) for every closed $V \subseteq V(k)$, there is an A-mod M s.t. $V(M) = V$~~

(4) $V_A(M \otimes N) \subseteq V_A(M) \cap V_A(N)$ (by standard arguments e.g. Feldvoss-W '10)

and equality holds under some restrictions -

on A : $\quad V_A(M\otimes N) = V_A(M) \cap V_A(N)$ if A is cocommutative (Friedlander-Pevtsova '07)
or "quantum elem. abel. gp" (Pevtsova-W '09)

on N : $\quad V_A(M\otimes N) = V_A(M) \cap V_A(N)$ if N is a "Carlson L_ζ-module"
(PW '09, FW '10)

<u>Open question</u> ~~Does~~ For what A is it always true that $\hat{V}(M\otimes N) = \hat{V}(M) \cap \hat{V}(N)$? ~~whenever A is almost cocommutative?~~
"tensor product property" (*)
~~(explain, e.g. $\mathbb{Z}$?)~~
~~(no known counterexample)~~
((or at least can we answer this question when $A = u_q(\mathfrak{g})$?))

(*) does not always hold! —

Julia Plavnik is here & on job market →

<u>Thm</u> (Plavnik-W) Benson-W (we are still working under that assumption) ~~Let A be a f.d. nonsemisimple Hopf algebra such that $H^*(A)$ is fin. gen. and $H^*(M)$ is fin. gen. as $H^*(A)$-mod. for all fin. dim. A-mods M.~~ If A (nonss.) satisfies (*) ~~the tensor product property~~, then A is a subalgebra of a Hopf algebra that does <u>not</u> satisfy (*) ~~#~~.

(Explain idea - ~~wreath~~-type coproduct so that t.p. of mods behaves like a semidirect or wreath product of groups)

<u>Open Question</u> Does (*) hold for all "quasitriangular" Hopf algebras?

[illegible]

<u>Related open questions</u> (maybe can be answered without support varieties)

Fix A.

(1) $M\otimes N$ projective iff $N\otimes M$ projective? $(\forall M, N)$ (implied by t.p. property: $V(M\otimes N) = V(M)\cap V(N) = V(N\otimes M)$ is 0 iff $M\otimes N$ is proj iff $N\otimes M$ is)

(2) M projective iff $M^{\otimes n}$ projective? $(\forall M)$ (implied by t.p. property, similarly)

(If time)

Examples (Benson - W)

let L be a finite group acting on G by automorphisms, notation ${}^{\ell}g$ $(\ell \in L,\ g \in G)$

$k^L := \mathrm{Hom}_k(L, k)$, an algebra under pointwise multiplication

$A := kG \ltimes k^L$ semidirect coproduct (dual to semidirect product $k^G \rtimes L$)

(a Hopf algebra)

tensor product of modules is not commutative up to iso ($\text{mod-}A \cong (\text{mod-}kG) \rtimes L$)

and $V(M \otimes N) \neq V(M) \cap V(N)$ in general

(see next page for details if there is more time)

Reference

"Varieties for modules of finite dimensional Hopf algebras" (W.)

www.math.tamu.edu/~sjw/pub/SuppVarSurvey-revised.pdf

Example: Noncommutative tensor products (joint work with Dave Benson)

Let L be a finite group acting on G by automorphisms, notation ${}^{\ell}g$ ($\ell \in L, g \in G$)

$k^L := \mathrm{Hom}_k(L, k)$, an algebra under pointwise multiplication,
basis $\{p_\ell \mid \ell \in L\}$ dual to L, so $p_h p_\ell = \delta_{h,\ell}\, p_\ell$ $\forall h, \ell \in L$

$A := kG \otimes k^L$ as an algebra (tensor product multiplication)

A is a Hopf algebra with "semidirect comultiplication" i.e. $\Delta: A \to A \otimes A$ given by

$$\Delta(g \otimes p_\ell) := \sum_{\substack{x,y \in L \\ xy = \ell}} (g \otimes p_x) \otimes ({}^{x^{-1}}\!g \otimes p_y)$$

$$\text{and } \quad \varepsilon(g \otimes p_\ell) = \delta_{1,\ell}, \qquad S(g \otimes p_\ell) = {}^{\ell^{-1}}(g^{-1}) \otimes p_{\ell^{-1}}$$

(Rk: A^* is dual, as a Hopf alg, to the skew gp alg $k^G \rtimes L$
with tensor product coalg. str.)

A-modules: The p_ℓ are central idempotents so if M is an A-mod then

$$M \cong \bigoplus_{\ell \in L} \big(\underbrace{M_\ell}_{\substack{\text{some} \\ kG\text{-mod}}} \otimes \underbrace{k p_\ell}_{1\text{-dim } k^L\text{-mod}} \big)$$

Tensor products: If M, N are A-mods then

$$(M \otimes N)_\ell \cong \bigoplus_{\substack{x,y \in L \\ xy = \ell}} M_x \otimes {}^{x}(N_y)$$

where ${}^{x}(N_y)$ is the kG-mod N_y but with action $g \cdot_x v = {}^{x^{-1}}(g)\, v$ $\forall v \in N_y, g \in G$

That is, the tensor category of A-modules has a semidirect product str.
$\text{mod-}A \cong$ "$kG\text{-mod} \rtimes L$"

There are L, G, M, N for which $V(M \otimes N) \neq V(M) \cap V(N)$
(in fact neither containment holds!)

(Be prepared to discuss more details, e.g. exactly what is $V(\,)$,
choices of L, G, M, N, etc., from other talk notes.)