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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

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Speaker's Name: Inna Entova-Aizenbud

Talk Title: Deligne categories and complexes of representations of symmetric groups

Date: 02 / Feb / 18 **Time:** 02 : 15 **am / pm (circle one)**

List 6-12 key words for the talk: representation theory, groups, symmetric algebras, symmetric group, Deligne categories, stablization

Please summarize the lecture in 5 or fewer sentences: We consider the multiplication map $Sym(V) \otimes V \to V$ as a complex of $GL(V)$-representations of length 2. We describe how tensor powers of the above complex define interesting complexes of representations of the symmetric group S_n. We explain how computing the cohomology of these complexes helps establish a relation between the Deligne categories and the representations of S_{∞}. (Joint work with D. Barter and Th. Heidersdorf)

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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Deligne cat. & Complexes of repr. of symm. grps

Base field: $\mathbb{C}$

~~Characters~~

Symmetric grps repr.: $S_n \circlearrowright \mathbb{C}$, $\mathfrak{h}_n = \mathbb{C}^n$

$\otimes$, semisimple: simples $\longleftrightarrow$ partitions of n:

$(\lambda_1, \dots, \lambda_k, \dots)$

$\lambda_1 \geq \lambda_2 \geq \dots$ $\sum \lambda_i = n$

$\mathfrak{h}_n = \underbrace{\boxed{}}_{}$ reflection $\oplus$ $\boxed{}$ trivial

Stabilization of representations of symm. grps:

Goal: connect 2 different settings.

Study sequences $(V_n \circlearrowleft S_n)$ s.t.

numerical invariants of repr. ~~become~~ are (eventual) polynomials in n as $n \gg 0$.

These occur in various contexts, such as

(Church Ellenberg Farb) • i-th cohomology$/\mathbb{Q}$ of conf. space ~~of~~ of n pts (ordered distinct) on a conn. oriented manifold M

• Space of poly. of deg. i on $X_r^{(n)}$ (rank variety of $n \times n$ matrices with cond. $rk(A) \leq r$ $\rightarrow$ can generalize!)

Example: Let λ be a partition of arbitrary size. Can consider

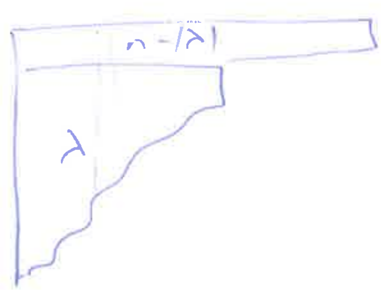
$$V_\lambda := \left(\lambda[n] \circlearrowleft S_n \right)$$

where

$$\lambda[n] := \begin{cases} (n-|\lambda|, \lambda_1, \lambda_2, \dots) & n \geq |\lambda| + \lambda_1 \\ 0 & \text{else} \end{cases}$$

$$\underbrace{\text{then} \quad \dim}_{\text{for } n \geq |\lambda|+\lambda_1} \lambda[n] = \frac{n!}{\prod_{(i,j)\in\lambda} h_\lambda(i,j) \cdot \prod_{1 \leq j \leq n-|\lambda|} (\lambda_j^\vee + 1 + n - |\lambda| - j)} =$$

$$= \frac{n!}{\prod_{(i,j)\in\lambda} h_\lambda(i,j) \prod_{1\leq j \leq n-|\lambda|} (n - |\lambda| - j + \lambda_j^\vee + 1)} =$$



$$= \frac{1}{\underset{\text{as a repr. of } S_{|\lambda|}}{\dim \lambda}} \cdot \left(\begin{array}{l} \text{polynomial in } n \text{ whose roots are} \\ \geq 0 \text{ integers} \notin \{ |\lambda| + j - \lambda_j^\vee - 1 \mid j \geq 1 \} \end{array} \right)$$

(note: for $k \geq |\lambda|+1$, $\lambda_k^\vee = 0 \Rightarrow$
$k + |\lambda| - 1 \in \{ |\lambda| + j - \lambda_j^\vee - 1 \mid j \geq 1 \} \Rightarrow$ polynomial of $\deg \leq 2|\lambda|$)

Natural settings for study of such sequences:

0) FI-modules (Church, Ellenberg, Farb):

FI = category of finite sets & injective maps (in particular, $\mathrm{End}([n]) \cong S_n$)

FI-mod = functors $FI \to \mathrm{Vec}_{f.d.}$

<u>objects</u>: sequences $(V_n \circlearrowleft S_n)$ of "compatible" representations (compatibility conditions given by S_n-equivariant maps $V_n \to V_{n+1}$)

<u>maps</u>: compatible maps of equivar. morphisms

- FI-modules satisfy the polynomiality conditions we mentioned

(<u>Thm</u> (CEF): FI-modules $V=(V_n \circlearrowleft S_n) \Rightarrow$ for $n \gg 0$, $\text{character}_{S_n}(V_n)(\sigma)_{\sigma \in S_n}$ = polynomial in (#i-cycles in σ) $i \leq r$ ← fixed, fixed)

1) <u>$\mathrm{Rep}(S_\infty)$</u> $\approx$ "infinite tails"

Consider the category of $\left. \text{FI-modules} \middle/ \text{Serre subcat of "finite" sequences} \right.$

This category can be seen as a category of repr. of the <u>infinite</u> symm. grp $S_\infty = \bigcup_{n\to\infty} S_n$.

$\mapsto$ <u>Algebraic (rational) repr. of S_∞</u>:

subquotients of $\bigoplus_{\text{finite}}$ of finite $\otimes$ powers of $\mathfrak{h}_\infty = \mathbb{C}\{e_1, e_2, \dots\}$

- This is an abelian $\underbrace{\text{symmetric monoidal}}_{}$ ($\otimes$) category, <u>not</u> s-s.
  SM $\leftrightarrow$ "comm. unital ring" category (additional <u>data</u>!)
- simples $\longleftrightarrow$ all partitions (any size)
  $L(\lambda) \longleftrightarrow \lambda$

(enough injectives: powers of $\mathfrak{h}_\infty$)

<u>Example</u>: $0 \to \underset{\text{reflection}}{L(\square)} \to \mathfrak{h}_\infty \to \mathbb{C} \to 0$

$e_i \mapsto 1$

<u>doesn't split</u>

Connection to S_n:

functors $\Gamma_N : \underline{Rep}(S_\infty) \xrightarrow{(\cdot)^{S_N^+}} Rep(S_N)$

where $S_\infty = \left(\begin{array}{c|c} S_N & \\ \hline & S_N^+ \end{array}\right)$

- Γ_N is $\otimes$, left-exact
- $L(\lambda) \longmapsto \lambda[N]$

N.B.:

$$\begin{tikzcd} \text{FI-mod} \arrow[r, "(V_n)\mapsto V_N"] \arrow[d, "q"'] & Rep(S_N) \\ \underline{Rep}(S_\infty) \arrow[ur, "\Gamma_N"'] & \end{tikzcd}$$

(quotient) q

each FI-mod M satisfies: $(M_n \circlearrowleft S_n)$

$$\Gamma_N\, q(M) = M_n \circlearrowleft S_N$$

for $N \gg 0$.

Hence have $\Gamma_N \circ q$ as direct summand of $\mathrm{Res}: \text{FI-mod} \to Rep(S_N)$ with isom. for large N (for specific FI-module M).

2) Deligne categories

- Idea: interpolate numerical invariants (polynomial!) to $n = t \in \mathbb{C}$

Deligne cat $\underline{\operatorname{Rep}} S_t$ (a.k.a. $\underline{\operatorname{Rep}}^{ab}(S_t)$)
$t \in \mathbb{C}$

- rigid symm. monoidal abelian category
  - symm. monoidal: $\otimes$, $\mathbb{1}$, commutativity
  - rigid: $\forall X\ \exists X^*$ (objects behave like f.d. vect. spaces)

maps $\mathbb{1} \longrightarrow X \otimes X^*$, $X^* \otimes X \longrightarrow \mathbb{1}$

Feature: allow us to define $\dim(X) \in \operatorname{End}(\mathbb{1})$

In our cat., $\dim(X) \in \operatorname{End}(\mathbb{1}) = \mathbb{C}$

$\Rightarrow$ $\dim(X)$ is a complex number ("polynomial in t for nice X")

~~abelian~~ ~~simples~~ ex: $\forall t$, have $\mathfrak{h}_t \in \underline{\operatorname{Rep}} S_t$, $\dim \mathfrak{h}_t = t$
("perm. repr")

- simples $\longleftrightarrow$ all partitions
L_λ ~~…~~ $\longleftrightarrow$ λ

Example: have map $\mathfrak{h}_t \twoheadrightarrow \mathbb{1} = \mathbb{1}_\phi$, $\uparrow$ dim t

for $t \neq 0$, this splits as $\mathfrak{h}_t \simeq \mathbb{1} \oplus \tilde{L}_\square$

for $t=0$, $\mathfrak{h}_t$ has Loewy filtration

$$\mathbb{1}, \tilde{L}_\square, \mathbb{1}$$

(hence $\dim \tilde{L}_\square = \begin{cases} t-1 & t \neq 0 \\ -2 & t=0 \end{cases}$)

- $\boxed{\text{not repr. of } \underline{\text{any grp}} !}$
- objects = subquotients of finite $\oplus$ of finite $\otimes$ powers of $\mathfrak{h}_t$.

- connection to S_n:

for $t = n \in \mathbb{Z}_{\geq 0}$, $\underline{\mathrm{Rep}}(S_t)$ is ~~...~~ a h.w. cat. (w/ infinitely many weights)

$\rightsquigarrow$ standards $\tilde{M}_\lambda$, costandards, tilting $\tilde{T}_\lambda$, proj "inj".

Then

$$\begin{array}{ccc} \tilde{T}_\lambda & \text{Karoubian Tilt } \underline{S}_{t=n} & \xhookrightarrow[\otimes, \oplus, \text{full}]{} \underline{\mathrm{Rep}}(S_t)_{t=n} \\ \downarrow & \downarrow {\scriptstyle \otimes, \oplus, \text{ess. surj. \& full}} & \\ \lambda[n] & \mathrm{Rep}\, S_n & \end{array}$$

Connecting these 2 settings:

Thm: (Barter, E., Heidersdorf)

$\forall t \in \mathbb{C}$

$\exists$ functor

$$\Gamma_t : \mathrm{Rep}(S_\infty) \longrightarrow \mathrm{Rep}(\underline{S}_t)$$
$$h_\infty \longmapsto h_t$$

s.t.

- Γ_t is exact, $\otimes$, faithful
- $\Gamma_t(L(\lambda)) = \tilde{M}_\lambda$
- Γ_t takes injectives to tilting

("analogue of induction from Borel subalgebra")

Idea:

- $\mathrm{Rep}(S_\infty)$ gen. by $\mathfrak{h}_\infty$
  $\Rightarrow$ constructing a left exact $\otimes$ functor based on $\mathfrak{h}_\infty \mapsto \mathfrak{h}_t$ is not difficult. (Sam-Snowden)

- injectives $\longmapsto$ tilting
  $\parallel$ $\parallel$
  direct summands in $\bigoplus_{\text{finite}} \mathfrak{h}_\infty^{\otimes r_i}$ ⟼ direct summands in $\bigoplus_{\text{finite}} \mathfrak{h}_t^{\otimes r_i}$

- Tricky: proving exactness & faithfulness.

- Idea: give a nice description of injective resolutions in $\mathrm{Rep}(S_\infty)$.

~~For this, consider~~

For this, consider the following setting:

Let
$$\Delta^k_N := \mathbb{C}\,\mathrm{Inj}(\{1,\dots,k\},\{1,\dots,n\})$$
$$\bar\Delta^k_N = \Delta^k_N \otimes \mathrm{sgn}_k \qquad S_n \times S_k$$

and maps

$$l_i : \{1,\dots,k\} \to \{1,\dots,k+1\} \qquad 1\le i\le k+1$$
(monotone, skipping i)

and
$$\mathrm{res}_i : \Delta^{k+1}_N \to \Delta^k_N,\quad f \mapsto f\circ l_i, \qquad \overline{\mathrm{res}}_i : \bar\Delta^{k+1}_N \to \bar\Delta^k_N$$

consider the complex

$$\underset{S_n\times S_N}{K^\bullet_{n,N}} = \dots \to \underset{\deg -k}{(\Delta^k_N \otimes \Delta^k_n)^{S_k}} \xrightarrow{\sum(-1)^i \mathrm{res}_i \otimes \overline{\mathrm{res}}_i} (\Delta^{k-1}_N \otimes \Delta^{k-1}_n)^{S_{k-1}} \to \dots$$

- This is the total complex of the n-cube given by:

  (vertex $(a_1,\dots,a_n)$, $a_i\in\{0,1\}$ $\longleftrightarrow$ subset $P\subseteq\{1,\dots,n\}$)
  $|P|=k$

  - at vertex P, have space $\mathbb{C}\,\mathrm{Inj}(P,\{1,\dots,N\})$
  - differentials $P\to Q$ (differ by 1 elem) given by res_i maps

Consider corresponding objects

$\forall t \in \mathbb{C}, n \in \mathbb{Z}_{\geq 0}$

$\Delta_{n,t} \in \mathrm{Rep}(\underline{S}_t)$, $\Delta_{n,\infty} \in \mathrm{Rep}(S_\infty)$

(tilting!)

and complexes

$\circlearrowleft S_n \times \underline{S}_t \qquad \circlearrowleft S_n \times S_\infty$

$K^\bullet_{n,t}$, $K^\bullet_{n,\infty}$

tilting complex / injective complex

Then we have

<u>Prop</u>: $\left. \begin{matrix} K^\bullet_{n,t} \\ K^\bullet_{n,\infty} \end{matrix} \right\}$ are exact except in degree $-n$

$K^\bullet_{\lambda,t} := K^\bullet_{n,t} \otimes_{S_n} \lambda^\vee$ is a tilting resolution of $\tilde{M}_\lambda$

$K^\bullet_{\lambda,\infty} := K^\bullet_{n,\infty} \otimes_{S_n} \lambda^\vee$ is an injective resolution of $L(\lambda)$

<u>Cor</u>: Knowing $R^i\Gamma_N$, can compute $H^\bullet(K^\bullet_{n,N})$