RATIONALITY OF BLOCKS OF QUASI-SIMPLE FINITE GROUPS

NIAMH FARRELL

Joint work with Radha Kessar

Let ℓ be a prime, $k = \overline{\mathbb{F}}_\ell$, G a finite group. Form the group algebra,

$$kG = B_1 \times \cdots \times B_r.$$

Each B_i is a "block" – an indecomposable algebra, and a 2-sided ideal. This decomposition into algebra factors is unique.

Let A be a finite dimensional k-algebra. Then A is *defined over* a subfield F of k if there exists an F-algebra A_0 such that $A \cong k \otimes_F A_0$. A *basic algebra* of A is a k-algebra which is Morita equivalent to A and whose simple modules all have dimension 1.

Theorem 0.1 (F. 2017, F.-Kessar)**.** *Let A be a basic algebra of a block of kG for a quasi-simple finite group G. Then A is defined over $\mathbb{F}_{\ell^4}$.*

1. Connection to Donovan's Conjecture

A *defect group* D of a block B is a finite ℓ-subgroup of G which measures how far B is from being semisimple.

$$D \text{ trivial} \iff B \text{ semisimple}$$

Suppose we are given a finite ℓ-group P. Define

$$\mathbb{B}_P := \{\text{blocks of finite group algebras with defect groups isomorphic to } P\}.$$

Conjecture 1.1 (Donovan's Conjecture)**.** *There are finitely many Morita equivalence classes of blocks in $\mathbb{B}_P$.*

Theorem 1.2 (Kessar, 2004)**.** *Donovan's Conjecture holds if and only if the following two conjectures hold:*

Conjecture 1.3 (Weak Donovan's Conjecture)**.** *There exists a bound on the entries of the Cartan matrices of the blocks in $\mathbb{B}_P$ which depends only on $|P|$.*

Conjecture 1.4 (Rationality Conjecture)**.** *There exists a bound on the minimum positive integer a such that the basic algebras of the blocks of $\mathbb{B}_P$ are defined over $\mathbb{F}_{\ell^a}$, and this bound depends only on $|P|$.*

2. Using Galois Conjugation

Let $\sigma : kG \to kG$ denote the *Galois Conjugation map*,

$$\sigma\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g \in G} \alpha_g^\ell g$$

This map permutes the blocks of kG. $B \cong \sigma(B)$ as rings, but they may not even be Morita equivalent as k-algebras.

Definition 2.1. The *Morita Frobenius number* of B is the minimum positive integer a such that $\sigma^a(B)$ is Morita equivalent to B. Denoted by $mf(B)$.

Proposition 2.2 (Kessar, 2004). *The Morita Frobenius number of B is the minimum positive integer a such that the basic algebras of B are defined over $\mathbb{F}_{\ell^a}$.*

We work over an ℓ-modular system $(K, \mathcal{O}, k)$:

$\mathcal{O}$ = a discrete valuation ring with max. ideal $\mathfrak{m}$
$K = \mathrm{frac}(\mathcal{O}) \to \mathrm{char}(K) = 0$, assume K is "large enough"
$k = \mathcal{O}/\mathfrak{m} \to$ assume $\mathrm{char}(k) = \ell$, algebraically closed

Let B be a block of kG. Then $B = kGb$ where $b \in Z(kG)$ is a primitive idempootent. Let $\pi : \mathcal{O}G \to kG$ be induced by the quotient map. Then $\exists!$ primitive idempotent $\hat{b} \in Z(\mathcal{O}G)$ st $\pi(\hat{b}) = b$.

$\mathrm{Irr}(G) :=$ K-valued irreducible characters of G
$\mathrm{Irr}(B) := \{\chi \in \mathrm{Irr}(G) \,|\, \hat{b}V = V \text{for a } KG \text{ module } V \text{ affording } \chi\}$.

$$\to \mathrm{Irr}(G) = \bigcup_{B \text{ a block of kG}} \mathrm{Irr}(B)$$

We can fix an automorphism $\hat{\sigma} : K \to K$ and define an action of $\hat{\sigma}$ on $\mathrm{Irr}(G)$ by

$$^{\hat{\sigma}}\chi(g) = \hat{\sigma}(\chi(g)),$$

such that $\mathrm{Irr}(\sigma(B)) = \{^{\hat{\sigma}}\chi \,|\, \chi \in \mathrm{Irr}(B)\}$.

2.1. Facts about $\boldsymbol{mf}(B)$.

(1) If $\sigma(B) \cong B$ then $mf(B) = 1$.
(2) If $\exists\, \chi \in \mathrm{Irr}(B)$ such that $\chi(g) \in \mathbb{Q}$ for all $g \in G$, then $^{\hat{\sigma}}\chi = \chi \Rightarrow \sigma(B) = B \Rightarrow mf(B) = 1$.
(3) If B has cyclic defect $\Rightarrow$ the basic algebras of B are Brauer tree algebras $\Rightarrow$ they are defined over $\mathbb{F}_\ell$... i.e. $mf(B) = 1$.

3. Results

Let B be a block of kG.

Proposition 3.1 (F., 2017). *If G is an alternating or sporadic group, then $mf(B)$ = 1.*

Alternating groups: characters of symmetric groups are rational valued
Sporadic groups: GAP

3.1. Finite groups of Lie type.

Let p be a prime. Let $\mathbf{G}$ be a simple, simply connected algebraic group over $\overline{\mathbb{F}}_p$, $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism, and $G := \mathbf{G}^F$ the finite group of fixed points.

3.1.1. *Case 1: $\ell = p$.*

Proposition 3.2 (F., 2017). $mf(B) = 1$

By a result of Humphreys we know there are $1+|Z(G)|$ blocks in kG, one of trivial defect and the rest of full defect. Can construct explicit k-algebra isomorphism $B \cong \sigma(B)$.

3.1.2. *Case 1:* $\ell \neq p$. By results of Broué-Michel and Hiss, $B \mapsto$ a collection of pairs $(\mathbf{T}, \theta)$ such that

$\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$
$\theta \in \mathrm{Irr}\ (\mathbf{T}^F)$ is an ℓ' character
$R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ has an irreducible consituent in $\mathrm{Irr}(B)$.

Suppose $\theta = 1$. Then B is unipotent. Unipotent characters of classical groups are rational valued.

Proposition 3.3 (F., 2017). *$mf\,(B) = 1$ except for two cases of E_8 where $mf\,(B) \leq 2$.*

Suppose $\theta \neq 1$.

Theorem 3.4 (F.-Kessar). *$mf\,(B) \leq 4$. Moreover, if $\mathbf{G}$ is not of type E_8, then $mf\,(B) \leq 2$ and if $\mathbf{G}$ is of type A, B or C then $mf\,(B) = 1$.*

Can bound $mf(B)$ in terms of the order of θ

A result of Bonnafé-Dat-Rouquier(2017) allows us to reduce to a situation where the order of θ is small (≤ 6).

Theorem 3.5 (F.-Kessar). *Donovan's Conjecture holds for the blocks of kG where $G = SL_n(q)$ ($q = p^a$ and $p \neq \ell$).*