MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rebekah Palmer **Email/Phone:** palmer@temple.edu (443) 621-7023

Speaker's Name: Vera Serganova

Talk Title: Representations of algebraic supergroups

Date: 02 / Feb / 18 **Time:** 04 : 30 am / pm (circle one)

List 6-12 key words for the talk: representation theory, groups, superalgebras, duality, categorification, tensor categories

Please summarize the lecture in 5 or fewer sentences: Representation theory of Lie superalgebras was originally motivated by applications in physics and topology. In recent years, duality and categorification unraveled new connections of superalgebras with other branches of representation theory. This lecture is an introduction to the subject with emphasis on geometric methods and applications to tensor categories. We also formulate some open problems.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

"Super" ~ $\mathbb{Z}_2$-graded $\qquad$ $\mathbb{C}$ linear

SVect $\quad$ $\mathbb{Z}_2$-graded f. dim. vector spaces $\quad V = V_0 \oplus V_1$

symmetric monoidal category SVect_{fin} $\quad V\otimes W \to W \otimes V$

$v\otimes w \mapsto (-1)^{\bar v \bar w} w \otimes v$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "Id"] \arrow[rrr, bend right] & V\otimes V^* \arrow[r, "b"] & V^*\otimes V \arrow[r, "\text{pairing}"] & \mathbb{C} \end{tikzcd}$$

$\dim V = \dim V_0 - \dim V_1$

$X \in End(V)$

$$\begin{tikzcd} X : \mathbb{C} \arrow[r] \arrow[rrr, bend right, "\mathrm{str}"'] & V\otimes V^* \arrow[r, "b"] & V^* \otimes V \arrow[r] & \mathbb{C} \end{tikzcd}$$

$S(V) = "S"(V_0) \otimes "\Lambda"(V_1)$.

Def. affine alg. group

fin. gen. commutative

a Hopf s.algebra $\quad A = A_0 \oplus A_1$ $\qquad \Delta : A \to A \otimes A$

mult. law

$A/(A_1)$ is a usual Hopf algebra $\mathcal{O}(G_0)$ $\qquad (a\otimes b)(c\otimes d) = (-1)^{\bar b \bar c}\, ac\otimes bd$.

$A = \mathcal{O}(G)$ $\qquad \mathrm{Lie}\, G = \mathfrak{g} = \{ \mathrm{Der} : A \to \mathbb{C} \}$

$\mathfrak{g}$ is a Lie superalgebra : $\quad \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad [\ ,\ ]$

$\mathfrak{g}_0$, $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module $\qquad [a,b] = -(-1)^{\bar a \bar b}[b,a]$

$S^2\mathfrak{g}_1 \to \mathfrak{g}_0$ $\cdot$ $\mathfrak{g}_0$-inv. map.

satisfying $[[x,x],x] = 0$. $\qquad [a,[b,c]] = [[a,b],c] + (-1)^{\bar a \bar b}[b,[a,c]]$

Theorem (Koszul, Masuoka) • $\mathfrak{g}$ - Lie superalgebra, G_0 - alg. group, $\mathrm{Lie}\, G_0 = \mathfrak{g}_0$

$s : G_0 \to \mathrm{Aut}(\mathfrak{g}_1)$

$ds = ad_{\mathfrak{g}_0}$

• Rep G is equivalent to the category of $(\mathfrak{g}, G_0)$-modules

$\to$ Schur-Weyl duality and Deligne's theorem.

Lie superalgebras (examples) $\qquad V = \mathbb{C}^{m|n}$ $\qquad \dim V_0 = m$

$\dim V_1 = n$

$\mathfrak{gl}(m|n) = \mathrm{End}_{\mathbb{C}}(V)$

$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{matrix} A \in \mathfrak{gl}(m) \\ D \in \mathfrak{gl}(n) \end{matrix} \right\}$ $\qquad \mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}$

$[X,Y] = XY - (-1)^{\bar X \bar Y} YX$ $\qquad$ if $m = n$ $\quad \mathrm{str}\, Id_V = 0$

$\mathfrak{sl}(m|n) = \{ X \in \mathfrak{gl}(m|n) \mid \mathrm{str}\, X = 0 \}$ $\qquad 0 \to \mathbb{C} \to \mathfrak{sl}(n|n) \to \mathfrak{psl}(n|n) \to 0$

$$V = \mathbb{C}^{m|2n}$$

$\mathfrak{g} = osp(m|2n)$ $\quad B(v,w) = (-1)^{\bar v \bar w} B(w,v)$ non-deg. form.

$= \{ X \in \mathfrak{gl}(V) \mid B(Xv, w) + (-1)^{\bar X \bar v} B(v, Xw) \} = 0$

$\mathfrak{g}_0 = o(m) \oplus sp(2n)$. $\quad B(v,w) \neq 0 \Rightarrow \bar v = \bar w$

$\mathfrak{g} = p(n)$ $\quad V = \mathbb{C}^{n|n}$ $\quad B(v,w) \neq 0 \Rightarrow \bar v = \bar w$ $\quad \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ $\begin{array}{l} B^t = B \\ C^t = -C \end{array}$

~~simple s. algebra classified by Kac (77)~~

Rep G = Rep $(\mathfrak{g}, G_0)$ in the case when G_0 is a reductive group.

(1) $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M = S(\mathfrak{g}_1) \otimes M$ $\quad S(\mathfrak{g}_1)$ is finite dimensional

enough projective objects.

(2) $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M = \mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} (M \otimes \Lambda^{top} \mathfrak{g}_1)$, every ~~projecti~~ projective object is injective and vice versa.

The category Rep G is not semisimple in most cases, has infinite global dimension.

Characters.

Borel subgroups $B \subset G$, flag supermanifolds, $T \subset B \subset G$

maximal torus

Highest weights $\lambda \in \Lambda^+$ $\quad L(\lambda)$ simple objects

Several Borel subgroups which are not conjugate.

Example: $GL(1|2)$ $\quad \mathbb{C}^{1|0} \subset \mathbb{C}^{1|1} \subset \mathbb{C}^{1|2}$

$\mathbb{C}^{0|1} \subset \mathbb{C}^{1|1} \subset \mathbb{C}^{1|2}$

$\mathbb{C}^{0|1} \subset \mathbb{C}^{0|2} \subset \mathbb{C}^{1|2}$

Borel-Weil-Bott theorem (Penkov): For a generic highest weight λ (typical)

$H^i(G/B, \mathcal{O}_\lambda)$ is non-zero exactly in one degree in this degree is irreducible.

<u>Open question</u>: Desribe $H^i(G/B, \mathcal{O}(\lambda))$ for a general weight λ.

<u>Kac-Weyl character formula</u>: $\frac{D_1}{D_0} \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}$

<u>Stratification, atypicality degree and blocks</u>:

$\operatorname{Ext}^i(\mathbb{C}, \mathbb{C})$
$\cong H^i(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$
$\cong S(\mathfrak{g}_1^*)^G$

Restrict to the case $G = GL(m|n)$

<u>Rep G is a highest weight category</u>: $\Delta(\lambda) = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_0(\lambda) \otimes \mathbb{C}[x_1, \dots, x_p]$

[$P_{the\ defect}^{CS}$]

$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \right\}$. <u>Remark</u>. The same does not work for $osp(m|2n)$. not a h. w. category.

$X = \{ x \in \mathfrak{g}_1 \mid [x, x] = 0 \}$ $\cancel{X}$ is G_0-invariant algebraic variety (conical)

$[x, x] = 2x^2 = 0$

$\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$

$\mathfrak{g}_x = \operatorname{Ker} \operatorname{ad} x / \operatorname{Im} \operatorname{ad} x$ again a superalgebra with reductive $\mathfrak{g}_0$. ~~part~~

$\operatorname{rk} x = k$ (as a matrix) $\Rightarrow \mathfrak{g}_x \simeq \mathfrak{gl}(m-k|n-k)$

$X_k = \{ x \in X \mid \operatorname{rk} x \le k \}$

$M \subset \operatorname{Rep} G \qquad M_x \overset{\text{def}}{=} \operatorname{Ker} x_M / \operatorname{Im} x_M$ is a $\mathfrak{g}_x$-module

- DS_x : ~~[illegible]~~ $\operatorname{Rep} G \longrightarrow \operatorname{Rep} G_x$ is a symmetric monoidal functor between tensor categories
  $M_x \otimes N_x \cong (M \otimes N)_x$ and $(M^*)_x \cong (M_x)^*$

$X_M = \{ x \in X \mid M_x \neq 0 \}$

X_M is closed G_0-invariant subvariety of G

$\operatorname{Rep} G = \bigoplus \operatorname{Rep}^{\chi} G_{\xi}$

block decomposition

$\chi \in$ central character $Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$.

degree of atypicality of χ is k (at χ)

$M \in \operatorname{Rep}^{\chi} G$, if M is simple then $X_M \subset X_k$

<u>Proposition</u> (Duflo, S) $\operatorname{rk} x = k$

$DS_x \operatorname{Rep}^{\chi} GL(m|n) \longrightarrow \operatorname{Rep}^{\chi'} GL(m-k|n-k)$

at $\chi' = $ at $\chi - k$

Two blocks of the same atypicality degree are equivalent as abelian categories

Sergeev:

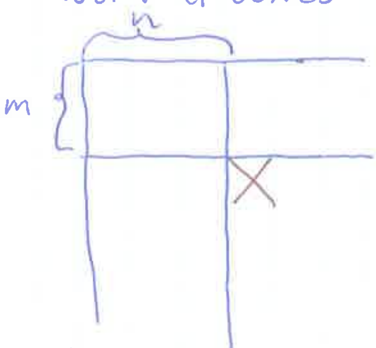
Let V be a s. vector space of dim $m|n$

$V^{\otimes d}$ has a natural actions of S_d

$s_{i\,i+1}$ $\quad V \otimes V \xrightarrow{\sigma} V \otimes V$

$$V^{\otimes d} = \oplus S'_\lambda(V)^{\oplus \dim Y_\lambda}$$

$\lambda \in \Gamma_{m,n}(d)$ the set of Young diagram with d boxes fitting into infinite m,n-hook



$(S'_\lambda(V) = \pi_\lambda V^{\otimes d}$ is called a Schur functor it can be defined on any sym. monoidal category)

<u>Theorem</u> (Deligne) Let $\mathcal{T}$ be a sym. monoidal rigid $\mathbb{C}$-linear category satisfying two conditions:

- $\mathcal{T}$ generated by finite family of objects $X_1, \dots, X_K$
- $\exists\ \lambda_1, \dots, \lambda_K$ (diagrams) $S_{\lambda_i}(X_i) = 0$

Then $\mathcal{T}$ is equivalent to the category $\mathrm{Rep}(G, \varepsilon)$, where G is an algebraic supergroup, $\varepsilon \in G_0$, $\mathrm{Rep}(G, \varepsilon)$ is the category $(\mathfrak{g}, G_0)$ modules V such that $\varepsilon(v) = (-1)^{\bar{v}} v$.

Let $k = \text{at}\, \mathcal{X}$ $\quad \mathrm{Rep}^{\mathcal{X}}\, GL(m|n) \longrightarrow \mathrm{Rep}^{\mathcal{X}'}\, GL(m-k \,|\, n-k)$

$\mathcal{X}'$ is already typical

M_x is a direct sum of several copies of a simple typical $\mathrm{Rep}\, GL(m-k|n-k)$ modules.

Blocks $\longleftrightarrow$ $(k,$ typical representation of $GL(m-k|n-k))$

Categorification (Brundan)

str XY defines an invariant form, a Casimir element

$\{x_i\}, \{y_i\}$ dual bases $\quad \sum_i x_i \otimes y_i$

V, V^* $\qquad \Omega: \; M \otimes V \longrightarrow M \otimes V$

$\qquad\qquad\quad M \otimes V^* \longrightarrow M \otimes V^*$

$E_i(M) = i$-th eigenspace of Ω ~~$\in \mathrm{End}$~~ $\{M \otimes V\}$

$F_i(M) =$ —— " —— of Ω in $M \otimes V^*$

e_i, f_i corresponding linear operators in

K_G, satisfy Serre's relation as operators of $sl(\infty)$

Thm. (Brundan)

- $K_G(\Delta) \simeq \Lambda^m E \otimes \Lambda^n E_*$, $\quad E, E_*$ st. and cost. repr. of $sl(\infty)$
- Block $\longleftrightarrow$ Weight spaces of $sl(\infty)$-mod.
- $\{P(\lambda)\}$ a canonical basis in the socle of $\Lambda^m \otimes \Lambda^n E_*$.

  $\uparrow$ indecomp. projectives

— Brundan - Stroppel $\quad$ parabolic category $\mathcal{O}_{gl}(m|\infty)$

Category $\mathcal{G}$ $\qquad\qquad\qquad\qquad\qquad\quad \mathcal{O}_{gl}(m+\infty)$.

Universal tensor category (Entova-Aizenbud, Hinich, S.)

$$\begin{array}{c} \mathrm{Rep}\, GL(m|n) \\ \downarrow D_x \quad \mathrm{rk}\, x = 1 \\ \mathrm{Rep}\, GL(m-1|n-1) \\ \downarrow \\ \mathrm{Rep}\, GL(m-2|n-2) \end{array}$$

$$\mathrm{Rep}\, \underline{GL}_t = \varprojlim_{m-n=t} \mathrm{Rep}\, GL(m|n)$$

$\mathrm{Rep}\, \underline{GL}_t$ is a h. w. monoidal rigid category, generated by V_t, V_t^*. Universal.

Theorem. (EHS) Let $\mathcal{T}$ be a symmetric monoidal category with rigid object X, $\dim X = t \in \mathbb{Z}$. Then

(a) if $S_\lambda(X) = 0$ for some partition λ, $\exists$ m, n, $m-n=t$, $\exists$ unique up to isomorphism SM faithful functor : $\mathrm{Rep}\, GL(m|n) \to \mathcal{T}$

$$V \mapsto X$$

(b) if $S_\lambda(X) \neq 0$ for all λ, then $\exists!$ SM faithful functor

$$\mathrm{Rep}\, \underline{GL}_t \to \mathcal{T}$$

$$V_t \mapsto X.$$