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Negative correlation and Hodge-Riemann relations

X smooth projective variety over $\mathbb{K} = \overline{\mathbb{F}}_p$

$A(X) = \dfrac{\text{algebraic cycles}}{\mathbb{Q}\text{-coefficients}} \Big/ \text{homological equivalence}$

$L \in A^1(X)$ hyperplane class

Conjecture: We should have

"PD" $\quad d = \dim(X)$

(Poincare Duality)

$$A^k(X) \times A^{d-k}(X) \longrightarrow \mathbb{Q}, \quad (x, y) \longmapsto \deg(x \cdot y)$$

is non degenerate for all k.

"HL"

(hard Lefschetz)

$$A^k(X) \longrightarrow A^{d-k}(X), \quad x \longmapsto L^{d-2k} x$$

is bijective for $k \leq d/2$.

"HR"

(Hodge-Riemann)

$$A^k(X) \times A^k(X) \longrightarrow \mathbb{Q}, \quad (x_1, x_2) \longmapsto (-1)^k \deg(L^{d-2k} x_1 x_2)$$

is positive definite on the kernel of L^{d-2k+1} for all $k \leq d/2$

V = vector space of dimension d over $K = \overline{\mathbb{F}_p}$.

E : spanning set of vectors in V, denoted by $\{1, 2, \dots, m\}$.

$\mathcal{L}$ = subspaces spanned by subsets of E.

$$= \bigsqcup_{k=0}^{d} \mathcal{L}_k$$

$\mathcal{L}_k$: collection of k-dimensional subspaces.

$$B(\mathcal{L}) = \bigoplus_{k=0}^{d} B^k(\mathcal{L}), \quad B^k(\mathcal{L}) = \bigoplus_{S \in \mathcal{L}_k} \mathbb{Q}\, x_S$$

graded $\mathbb{Q}$-algebra with the following multiplication:

$$x_{S_1} \cdot x_{S_2} = \begin{cases} x_{S_1 + S_2} & \text{if } S_1 \cap S_2 = 0 \\ 0 & \text{if otherwise.} \end{cases}$$

$L = x_1 + \dots + x_m$ = sum of all lines in $\mathcal{L}$

$$= \sum_{S \in \mathcal{L}_1} x_S \in B^1(\mathcal{L}).$$

$$B^d(\mathcal{L}) \simeq \mathbb{Q}, \quad x_V \longmapsto 1$$

we denote this isomorphism by "deg".

$$\deg(L^d) = \deg(x_1 + \dots + x_m)^d = d!\ \#\text{bases of } V \text{ in } E$$

$\deg(x_i L^{d-1}) = (d-1)!$ # of bases of V in E containing i.

$\deg(x_i x_j L^{d-2}) = (d-2)!$ # of bases of V in E containing i and j.

Def: A matroid M on $\{1, 2, \dots, n\} = E$ is $\mathcal{B} \subseteq 2^E$ such that:

For any $B_1, B_2 \in \mathcal{B}$, $x_1 \in B_1 \setminus B_2$, there is $x_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus x_1) \cup x_2 \in \mathcal{B}$.

Ex: ① G = connected graph E = set of edges
$\mathcal{B}$ = spanning trees of G.

② V = vector space over $\mathbb{K}$
E = spanning set of vectors in V
$\mathcal{B}$ = bases of V in E.

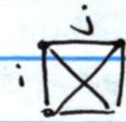
Def: For a matroid M on $E = \{1, 2, \dots, n\}$ define a $m \times m$ matrix $HR(M)$ by

$$HR(M)_{ij} = \begin{cases} 0 & \text{if } i = j \\ b_{ij} & \text{if } i \neq j \end{cases},$$

where b_{ij} = # of bases of M containing i and j.

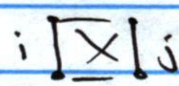
Thm (H.-Wang): For any matroid M, $HR(M)$ has exactly one positive eigenvalue.

Ex: $G =$  $= K_4$



$b_{ij} = 3$

$$HR(M) = \begin{pmatrix} 0 & 3 & 3 & 3 & 3 & 4 \\ 3 & 0 & 3 & 3 & 4 & 3 \\ 3 & 3 & 0 & 4 & 3 & 3 \\ 3 & 3 & 4 & 0 & 3 & 3 \\ 3 & 4 & 3 & 3 & 0 & 3 \\ 4 & 3 & 3 & 3 & 3 & 0 \end{pmatrix}$$



$b_{ij} = 4$

eigenvalues: $16, -2, -2, -4, -4, -4$.

Negative Correlation

G = finite connected graph
T = random spanning tree

"For any edges $i \neq j$, $\Pr(i \in T) \geq \Pr(i \in T \mid j \in T)$"

In other words, $\dfrac{b_i}{b} \geq \dfrac{b_{ij}}{b_j}$,

for b = # spanning trees
b_i = # spanning trees containing i
b_{ij} = " i and j

Q: Does this hold for arbitrary matroid M?

Yes, if M is realizable over every field.

In 1974, Seymour and Welsh found
M over $\mathbb{F}_2$ with $\dfrac{b\, b_{ij}}{b_i\, b_j} = 1.02\ldots$ for some i, j.

Obs: $HR(M) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

Restrict $HR(M)$ to $\mathrm{span}(e_i, e_j, e_1 + \cdots + e_n)$ to get a 3×3 matrix $HR_{ij}(M)$.

Applying Cauchy interlacing, we see that $HR_{ij}(M)$ has exactly one positive eigenvalue.

$$\longrightarrow \det HR_{ij}(M) \geq 0 \quad \longleftrightarrow \quad 2 \geq \left(\frac{\operatorname{rank}(M)}{\operatorname{rank}(M)-1}\right) \frac{b\, b_{ij}}{b_i\, b_j}$$

Conclusion: For any matroid M, elements i, j,

$$\frac{b\, b_{ij}}{b_i\, b_j} < 2$$

Question (Correlation constant of a field):
How large can $\frac{b b_{ij}}{b_i b_j}$ over a given field be?

Current record (Benjamin Schröter):

1.14 ... for $\mathbb{F}_2$

1.07 ... for $\mathbb{Q}$