A counterexample to the extension space conjecture for realizable oriented matroids

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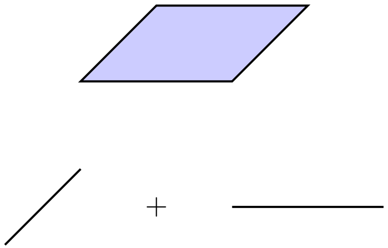
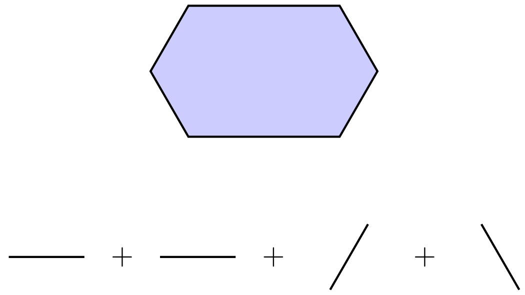
Statement of the conjecture

Zonotopes

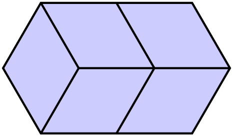
A *zonotope* is a Minkowski sum of line segments.

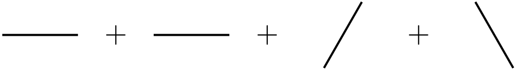




Zonotopal tilings

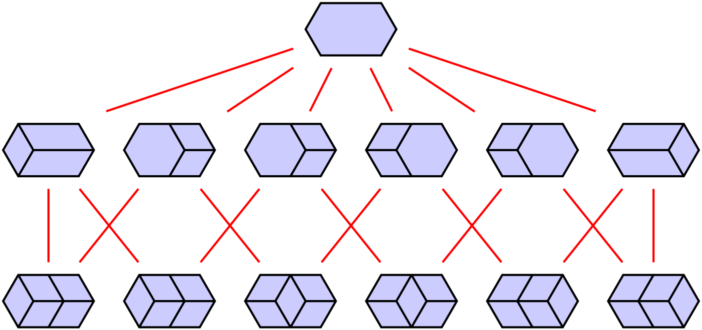
A *zonotopal tiling* of a zonotope Z is a tiling of Z with zonotopes generated by subsets of the generating vectors of Z.





The poset of zonotopal tilings

The set of all zonotopal tilings of a zonotope can be partially ordered by *refinement*.



A tiling is *proper* if it is not the maximum tiling.

The extension space conjecture

Conjecture (Sturmfels-Ziegler '93)

The poset of proper zonotopal tilings of a zonotope with n generators and dimension d is homotopy equivalent to a sphere of dimension $n - d - 1$.

Note: The "topology of a poset" refers to the topology of its *order complex*, which is the simplicial complex whose simplices are the chains of the poset.

The generalized Baues conjecture

We can make analogous conjectures for other classes of polytopal subdivisions. For example:

Question

Is the poset of all monotone cellular strings of a polytope homotopy equivalent to a sphere (of appropriate dimension)?

Question

Is the poset of all polyhedral subdivisions of a point set homotopy equivalent to a sphere (of appropriate dimension)?

The general framework is known as the *Generalized Baues problem*.

First question is yes (Billera-Kapranov-Sturmfels '94) and second question is no (Santos '06).

Relation to oriented matroids

Oriented matroids

Let X be a finite arrangement of vectors in $\mathbb{R}^d$. The *oriented matroid* of X is an object which encodes certain combinatorial information about the arrangement.

Example: WLOG X is full-dimensional in $\mathbb{R}^d$. For all x_1, $\ldots$, $x_d \in X$, the oriented matroid of X contains the data of

$$\mathrm{sign}(\det(x_1, \ldots, x_d)).$$

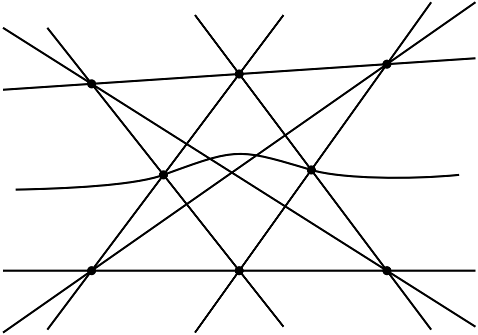
This data is called the *chirotope*, and is in fact enough to determine the oriented matroid.

Oriented matroids

In general, oriented matroids are abstractions of the above objects, and may or may not correspond to some real vector arrangement. Those that do are called *realizable*.

Topological representation of oriented matroids

There is a correspondence between oriented matroids and combinatorial classes of *pseudo-hyperplane arrangements*.

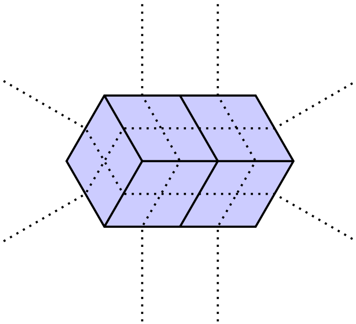


An oriented matroid is realizable iff it can be represented by a real hyperplane arrangement.

Zonotopes and oriented matroids

Theorem (Bohne-Dress '92)

Zontopal tilings can be mapped to oriented matroids.



Zonotopes and oriented matroids

Theorem (Bohne-Dress '92)

Zontopal tilings can be mapped to oriented matroids.

For a given zonotope, consider all zonotopal tilings corresponding to realizable oriented matroids which can be realized by hyperplanes normal to the generating vectors. These form a poset within the poset of all zonotopal tilings.

The extension space conjecture equivalently states that the poset of all zonotopal tilings of a zonotope can be deformation retracted onto the above smaller poset.

Realizable vs non-realizable?

Realizable oriented matroids are represented by hyperplane arrangements, and all oriented matroids are represented by tame deformations of hyperplane arrangements.

Question: If we have a "space of oriented matroids", can this space be deformation retracted onto one of its "realizable parts"?

Question: Realization spaces of realizable oriented matroids are arbitrarily bad (Mnëv). Do the non-realizable oriented matroids provide a way to "smooth" things out?

Positive results

Sturmfels and Ziegler ('93) proved the conjecture for the following zonotopes:

- Dimension at most 2
- $n - d \leq 3$
- Cyclic zonotopes

Bailey ('97) proved it for zonotopes whose generating sets contain at most $d + 1$ distinct vectors.

Stronger conjectures

The extension space conjecture is a special case of the following conjectures.

- The generalized Baues conjecture (Billera-Kapranov-Sturmfels '94), already disproven in general (Rambau-Ziegler '96) and in important cases (Santos '06).
- Combinatorial Grassmannian conjecture (MacPherson-Mnëv-Ziegler '93)

Counterexample

Main result

Conjecture

The poset of proper zonotopal tilings of a zonotope with n generators and dimension d is homotopy equivalent to a sphere of dimension $n - d - 1$.

Theorem (L.)

There exists a zonotope with $n > 4$ generators and dimension 3 whose poset of proper zonotopal tilings is disconnected.

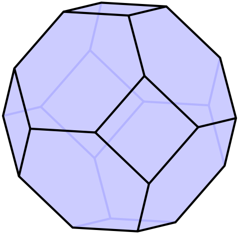
Therefore, the poset cannot be homotopy equivalent to a sphere of dimension $n - 4$.

Permutohedra

The d-dimensional *permutohedron* is the zonotope generated by the vectors

$$\{e_i - e_j : 1 \leq i < j \leq d + 1\}$$

where e_i is the i-th standard basis vector.



The construction

Let E_N be the multiset containing each element of

$$\{e_i - e_j : 1 \leq i < j \leq 4\}$$

repeated N times.

Let $\tilde{E}_N$ be the configuration obtained by perturbing each element of E_N by a small random displacement in the span of E_N.

Theorem

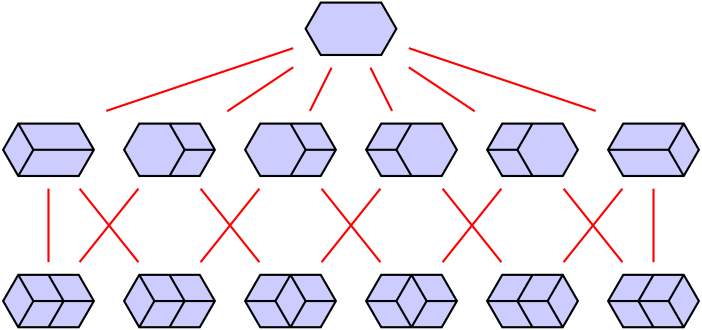
For large enough N, with positive probability, $\tilde{E}_N$ contains a subcollection (with size > 4) whose generated zonotope has disconnected poset of proper zonotopal tilings.

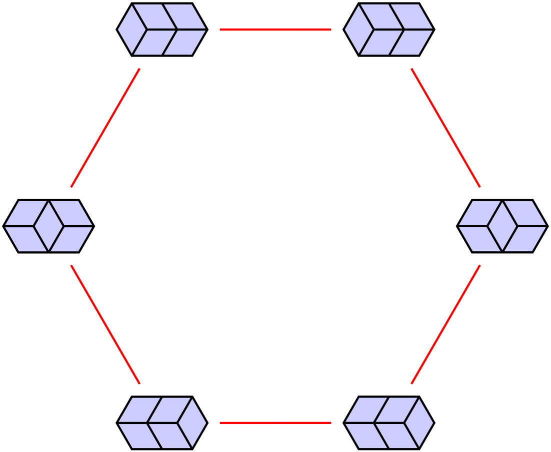
A preliminary result

Theorem (L.)

For $N > 100$, the zonotope generated by E_N has disconnected flip graph of zonotopal tilings.

The *flip graph* is the graph whose vertices are the minimal elements of the tiling poset and whose edges are the "minimal moves" (flips) between them.





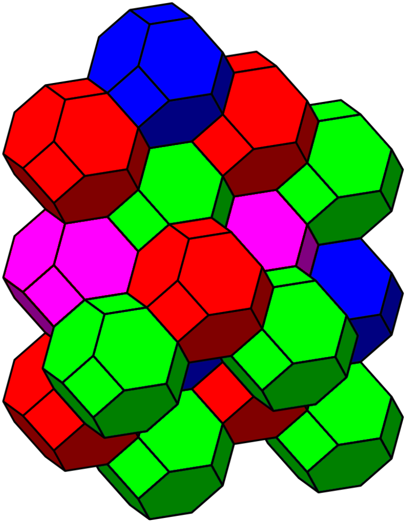
A preliminary result

Theorem (L.)

For $N > 100$, the zonotope generated by E_N has disconnected flip graph of zonotopal tilings.

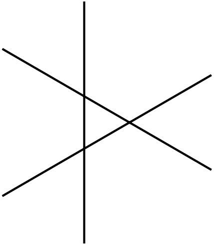
Proof.

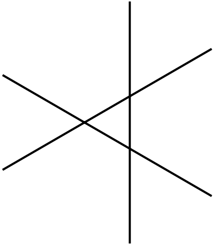
- The zonotope generated by E_N is a large permutohedron.
- Tile the middle of the large permutohedron with unit permutohedra.
- Tile the unit permutohedra randomly.*



Chirotopes

Each oriented matroid has an associated *chirotope* χ. χ is a function from tuples of pseudo-hyperplanes to the set $\{+, 0, -\}$ which tells you how the pseudo-hyperplanes are oriented with respect to each other.





$$\chi(a) = \pm 1$$

a is a 3-tuple containing the three pseudolines

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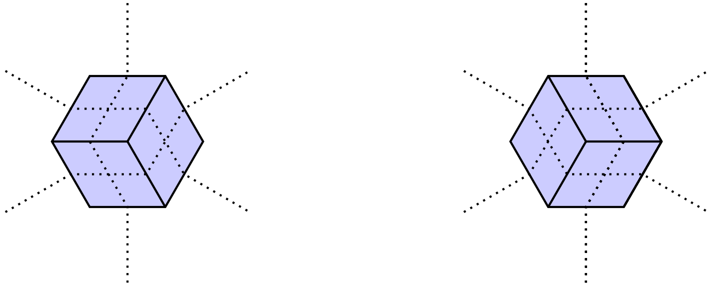


$$\chi(a) = \pm 1$$

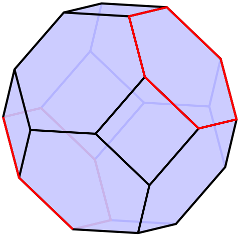
a is a 3-tuple containing the three pseudolines

Properties of chirotopes

The chirotope of an oriented matroid satisfies certain relations, which are abstractions of the Grassmann-Plücker relations.

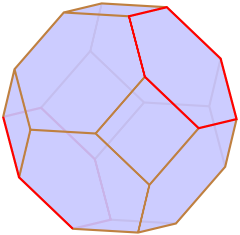
A flip on a zonotopal tiling changes the sign of the chirotope at some of the tuples.

Proof of flip non-connectivity



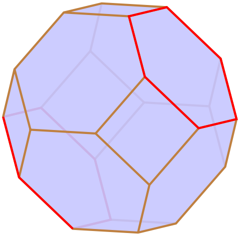
$\chi(a)$

Proof of flip non-connectivity



$\chi(a) \quad \chi(b) \quad \chi(c) \quad \chi(d)$

Proof of flip non-connectivity



$$\chi(a) = \chi(b) + \chi(c) + \chi(d)$$

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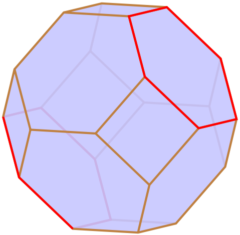

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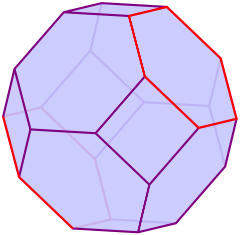

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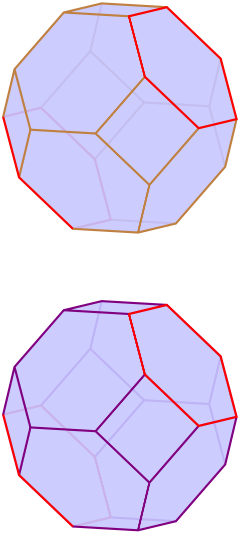
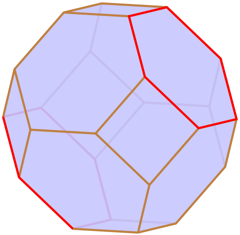
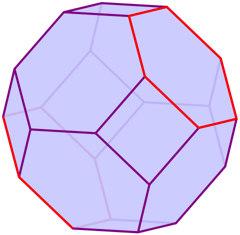


$$\chi(a) = \chi(b) + \chi(c) + \chi(d)$$
$$\chi(a) = \chi(b') + \chi(c') + \chi(d')$$

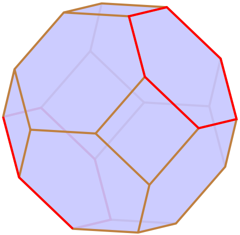
Proof of flip non-connectivity

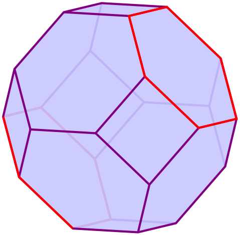




$$\chi(a) = \chi(b) + \chi(c) + \chi(d)$$
$$\chi(a) = \chi(b') + \chi(c') + \chi(d')$$
$$\chi(a) = \chi(b'') + \chi(c'') + \chi(d'')$$
$$\vdots$$

Proof of flip non-connectivity





$$\chi(a) = \chi(b) + \chi(c) + \chi(d)$$

$$\chi(a) = \quad + \quad + \quad + \quad + \quad +$$

$$\chi(a) = \chi(b'') + \chi(c'') + \chi(d'')$$

$$\vdots$$

How to tile randomly?

Restrict the tilings on the unit permutohedra to those of a certain form.

Observation: For any such tiling of a unit permutohedra, we can change the orientation of any two pairs of opposite hexagonal faces and obtain another tiling of the same form.

"Plane move": Take a plane in the arrangement and change the orientation of every hexagonal face contained in this plane simultaneously. Do this to a random subset of planes.

Sketch of proof of extension space conjecture

Theorem

There exists a zonotope with $n > 4$ generators and dimension 3 whose poset of proper zonotopal tilings is disconnected.

Follows from the following:

Theorem

For $N \gtrsim 10^5$, with positive probability, the zonotope generated by $\tilde{E}_N$ has disconnected flip graph of zonotopal tilings.

Proposition

If S is a set of vectors in general position whose zonotope has disconnected flip graph, then there is some non-trivial subset of S whose zonotope has disconnected poset of proper zonotopal tilings.

Further questions

- Related questions
  - Las Vergnas conjectures on existence of flips and connectivity of uniform oriented matroids under flips
  - Matroid Grassmannian (MacPhersonian) conjecture

- Smaller examples? Higher homology groups?

- Better understanding of random matroids/oriented matroids?

Thank you!