Recent developments on chromatic quasisymmetric functions

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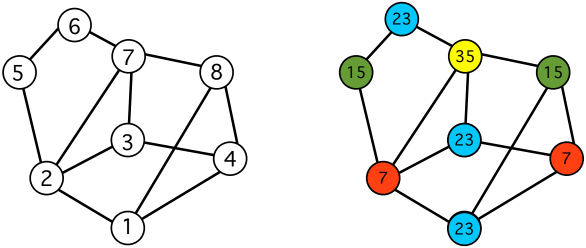
Chromatic symmetric functions

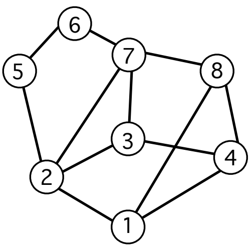


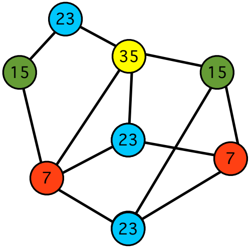
Let $C(G)$ be set of proper colorings of graph $G = ([n], E)$, where a proper coloring is a map $c : [n] \to \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E$.

Chromatic symmetric function (Stanley, 1995)

$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

Chromatic symmetric functions





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$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

$$X_G(\underbrace{1, 1, \ldots, 1}_{m}, 0, 0, \ldots) = \chi_G(m)$$

Chromatic symmetric functions

Let Π_G be the bond lattice of G.
Whitney (1932):

$$\chi_G(m) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) m^{|\pi|}$$

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Stanley (1995): Let p_λ denote the power-sum symmetric function associated with $\lambda \vdash n$. Then

$$X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)}(\mathbf{x}).$$

Equivalently

$$\omega X_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} |\mu(\hat{0}, \pi)| p_{\mathrm{type}(\pi)}(\mathbf{x}),$$

which implies that $\omega X_G(\mathbf{x})$ is p-positive.

b-Positivity

Important bases for the vector space Λ_n of homogeneous symmetric functions of degree n:

- complete homogeneous symmetric functions: $\{h_\lambda : \lambda \vdash n\}$
- elementary symmetric functions: $\{e_\lambda : \lambda \vdash n\}$
- power-sum symmetric functions: $\{p_\lambda : \lambda \vdash n\}$
- Schur functions: $\{s_\lambda : \lambda \vdash n\}$

Involution $\omega : \Lambda_n \to \Lambda_n$ defined by $\omega(h_\lambda) = e_\lambda$.

Let $b = \{b_\lambda : \lambda \vdash n\}$ be a basis for Λ_n. A symmetric function $f \in \Lambda_n$ is said to be b-positive if $f = \sum_{\lambda \vdash n} c_\lambda b_\lambda$, where $c_\lambda \geq 0$.

h-positive $\implies$ p-positive and Schur-positive.
f is e-positive $\iff$ ωf is h-positive.

e-positivity

$$X_{K_{3,1}} = 4e_4 + 5e_{3,1} - 2e_{2,2} + e_{2,1,1}$$

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- The incomparability graph $\mathrm{inc}(P)$ of a finite poset P on $[n]$ is the graph whose edges are pairs of incomparable elements of P.
- A poset P is said to be $(a + b)$-free if P contains no induced subposet isomorphic to the disjoint union of an a element chain and a b element chain.

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Conjecture (Stanley-Stembridge (1993))

If P is $(3+1)$-free then $X_{\mathrm{inc}(P)}$ is e-positive.

Stanley-Stembridge e-positivity conjecture

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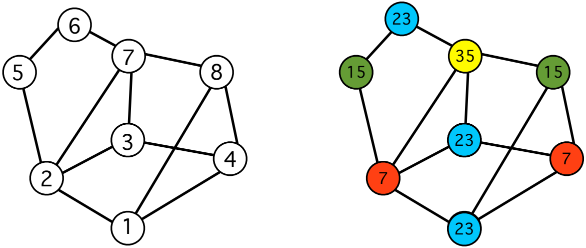
If P is $(3+1)$-free then $X_{\mathrm{inc}(P)}$ is e-positive.

- Haiman (1993): work on Hecke algebra characters $\implies$ $X_{\mathrm{inc}(P)}$ is Schur positive.
- Gasharov (1994): expansion in the Schur basis $\{s_\lambda : \lambda \vdash n\}$
- Chow (1996): expansion in Gessel's fundamental quasisymmetric function basis $\{F_\mu : \mu \models n\}$
- Guay-Paquet (2013): If true for unit interval orders (posets that are both (3+1)-free and (2+2)-free) then true in general i.e. for posets that are (3+1)-free.

Quasisymmetric refinement



Chromatic quasisymmetric function (Shareshian and MW)

$$X_G(\mathbf{x}, t) := \sum_{c \in C(G)} t^{\mathrm{des}(c)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

where

$$\mathrm{des}(c) := |\{\{i,j\} \in E(G) : i < j \text{ and } c(i) > c(j)\}|.$$

Quasisymmetric refinement

$G =$ 

$$X_G(\mathbf{x}, t) = e_3 + (e_3 + e_{2,1})t + e_3 t^2$$

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$$X_G(\mathbf{x}, t) = (e_3 + F_{1,2}) + 2e_3 t + (e_3 + F_{2,1})t^2$$

where $F_\mu(x_1, x_2, \dots) :=$ fundamental quasisymmetric function indexed by composition μ

Chromatic quasisymmetric functions that are symmetric

A natural unit interval order is a unit interval order with a certain natural canonical labeling.
Example: The poset $P_{n,r}$ on $[n]$ with order relation given by $i <_P j$ if $j - i \geq r$. Let

$$G_{n,r} := \mathrm{inc}(P_{n,r}) = ([n], \{\{i,j\} : 0 < j - i < r\})$$

When $r = 2$, $G_{n,r}$ is the path

$$1 - 2 - \cdots - n$$

and

$$X_{G_{n,r}} = \sum_{w \in W_n} t^{\mathrm{des}(w)} x_w,$$

where $W_n = \{w \in \mathbb{P}^n : \text{ adjacent letters of w are distinct}\}$.
These are called Smirnov words.

Chromatic quasisymmetric functions that are symmetric

Theorem (Shareshian and MW)

If G is the incomparability graph of a natural unit interval order then the coefficients of powers of t in $X_G(\mathbf{x}, t)$ are symmetric functions and form a palindromic sequence.

$$X_{G_{3,2}} = e_3 + (e_3 + e_{2,1})t + e_3 t^2$$
$$X_{G_{4,2}} = e_4 + (e_4 + e_{3,1} + e_{2,2})t + (e_4 + e_{3,1} + e_{2,2})t^2 + e_4 t^3$$

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Conjecture (Shareshian and MW - refinement of Stan-Stem)

If G is the incomparability graph of a natural unit interval order then the coefficients of powers of t in $X_G(\mathbf{x}, t)$ are e-positive and form an e-unimodal sequence.

True for

$r = 1, n$ (easy)

$r = 2, n-1, n-2$ (work of Shareshian and MW)

$1 < r < n \leq 9$ (computer)

Our approach - a bridge to Hessenberg varieties

Let G be a natural unit interval graph (i.e., the incomparability graph of a natural unit interval order).
Let $\mathcal{H}_G$ be the regular semisimple Hessenberg variety associated with G.
Tymoczko uses GKM theory to define a representation of $\mathfrak{S}_n$ on each cohomology $H^{2j}(\mathcal{H}_G)$.

Conjecture (Shareshian and MW (2012))

Let $\mathrm{ch} H^{2j}(\mathcal{H}_G)$ *be the Frobenius characteristic of Tymoczko's representation of* $\mathfrak{S}_n$ *on* $H^{2j}(\mathcal{H}_G)$*. Then*

$$\omega X_G(\mathbf{x}, t) = \sum_{j \geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G) t^j.$$

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If this conjecture is true then our refinement of the Stanley-Stembridge e-positivity conjecture is equivalent to

Conjecture

Tymoczko's representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_G)$ is a permutation representation for which each point stabilizer is a Young subgroup.

The bridge conjecture is true!

Let G be a natural unit interval graph.

Theorem (Brosnan and Chow (2015), Guay-Paquet (2016))

Let $\mathrm{ch} H^{2j}(\mathcal{H}_G)$ be the Frobenius characteristic of Tymoczko's representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_G)$. Then

$$\omega X_G(\mathbf{x}, t) = \sum_{j \geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G) t^j.$$

Combinatorial consequences:

- $X_G(\mathbf{x}, t)$ is Schur-positive and Schur-unimodal.
- Generalized q-Eulerian polynomials are q-unimodal.

Algebro-geometric consequences:

- Multiplicity of irreducibles in Tymoczko's representation can be obtained from an expansion of $X_G(\mathbf{x}, t)$ in Schur basis.
- Character of Tymoczko's representation can be obtained from an expansion of $X_G(\mathbf{x}, t)$ in power-sum basis.

Schur and power-sum expansions

Let $G = \mathrm{inc}(P)$ where P is a natural unit interval order.

For $\sigma \in \mathfrak{S}_n$, a G-inversion of σ is an inversion $(\sigma(i), \sigma(j))$ of σ such that $\{\sigma(i), \sigma(j)\} \in E(G)$. Let $\mathrm{inv}_G(\sigma)$ be the number of G-inversions of σ.

Theorem (Shareshian and MW, t=1 Gasharov)

$$X_G(\mathbf{x}, t) = \sum_{\lambda \vdash n} \sum_{T \in \mathcal{T}_{P,\lambda}} t^{\mathrm{inv}_G(w(T))} s_\lambda.$$

Consequently $X_G(\mathbf{x}, t)$ is Schur-positive.

Theorem (Athanasiadis, conjectured by Shareshian and MW)

$$\omega X_G(\mathbf{x}, t) = \sum_{\lambda \vdash n} \sum_{T \in \mathcal{R}_{P,\lambda}} t^{\mathrm{inv}_G(w(T))} z_\lambda^{-1} p_\lambda$$

Consequently $\omega X_G(\mathbf{x}, t)$ is p-positive.

Application: using stable principal specialization (Shareshian-W)

Define the generalized q-Eulerian polynomials

$$A_n^{(r)}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq r}(\sigma)} t^{\mathrm{inv}_{<r}(\sigma)},$$

where

$$\begin{aligned} \mathrm{inv}_{<r}(\sigma) &:= |\{(i,j) : 1 \leq i < j \leq n, \quad 0 < \sigma(i) - \sigma(j) < r\}| \\ \mathrm{maj}_{\geq r}(\sigma) &:= \sum_{i : \sigma(i) - \sigma(i+1) \geq r} i \end{aligned}$$

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Theorem (consequence of Schur-unimodality of $X_{G_{n,r}}(\mathbf{x},t)$)

$A_n^{(r)}(q,t)$ is palindromic and q-unimodal for all $r \in [n]$.

$$A_4^{(2)}(q,t) = 1 + (3+2q+3q^2+2q^3+q^4)t + (3+2q+3q^2+2q^3+q^4)t^2 + t^3$$

$$A_5^{(2)}(q,t) = 1 + (4+3q+5q^2+\dots)t + (6+6q+11q^2+\dots)t^2 + \dots$$

Problem: Find an elementary proof of q-unimodality.

A weakly increasing sequence $\mathbf{m} = (m_1, \ldots, m_n)$ of integers satisfying $1 \leq i \leq m_i \leq n$, will be called a Hessenberg sequence.

Let $\mathcal{F}_n$ be the set of all flags of subspaces of $\mathbb{C}^n$

$$F : F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$$

where $\dim F_i = i$.

Fix an $n \times n$ diagonal matrix D with distinct diagonal entries and let $\mathbf{m} = (m_1, \ldots, m_n)$ be a Hessenberg sequence.

The type A regular semisimple Hessenberg variety associated with $\mathbf{m}$ is

$$\mathcal{H}(\mathbf{m}) := \{F \in \mathcal{F}_n \mid DF_i \subseteq F_{m_i} \ \ \forall i \in [n]\}.$$

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Every natural unit interval graph G can be associated with a Hessenberg sequence $\mathbf{m}(G)$. Let

$$\mathcal{H}_G := \mathcal{H}(\mathbf{m}(G)).$$

GKM theory and moment graphs

Goresky, Kottwitz, MacPherson (1998): Construction of equivariant cohomology ring of smooth complex projective varieties with a torus action. From this, one gets ordinary cohomology ring.

The group T of nonsingular $n \times n$ diagonal matrices acts on

$$\mathcal{H}_G := \{F \in \mathcal{F}_n \mid DF_i \subseteq F_{m_i(G)} \ \forall i \in [n]\}.$$

by left multiplication.

Moment graph: graph whose vertices are T-fixed points and edges are one-dimensional orbits.

Fixed points of the torus action:

$$F_\sigma : \langle e_{\sigma(1)} \rangle \subset \langle e_{\sigma(1)}, e_{\sigma(2)} \rangle \subset \cdots \subset \langle e_{\sigma(1)}, \ldots, e_{\sigma(n)} \rangle$$

where σ is a permutation.
So the vertices of the moment graph can be represented by permutations.

Combinatorial description of the moment graph

For any natural unit interval graph $G = ([n], E)$, let $\Gamma(G)$ be the graph with vertex set $\mathfrak{S}_n$ and edge set

$$\{\{\sigma, \sigma(i,j)\} : \sigma \in \mathfrak{S}_n \text{ and } \{i,j\} \in E\}.$$

$\Gamma(G)$ is the moment graph for the Hessenberg variety $\mathcal{H}_G$.

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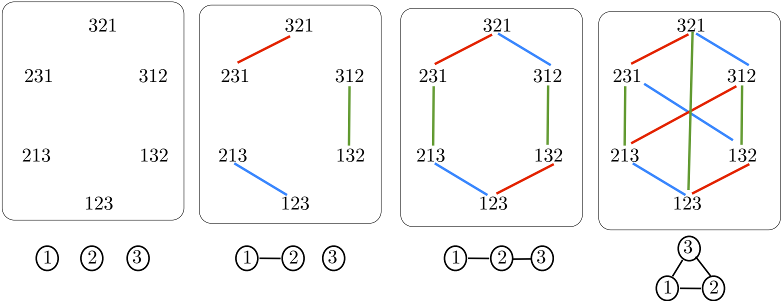
Example: $n = 3$.

Color coded edge labels: **(1,2)** **(2,3)** **(1,3)**



The equivariant cohomology ring $H_T^*(\mathcal{H}_G)$

$H_T^*(\mathcal{H}_G)$ is isomorphic to a subring of $R_n := \prod_{\sigma \in \mathfrak{S}_n} \mathbb{C}[t_1, \ldots, t_n]$.

For $p \in R_n$, let $p_\sigma(t_1, \ldots, t_n) \in \mathbb{C}[t_1, \ldots, t_n]$ denote the σ-component of p, where $\sigma \in \mathfrak{S}_n$.

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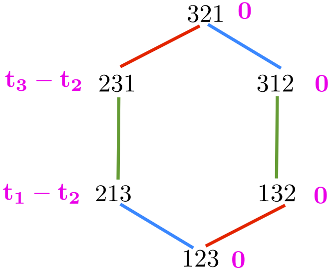

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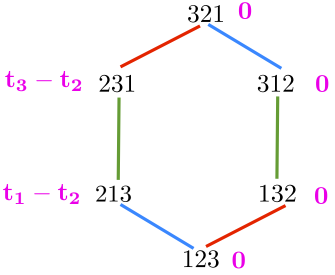

$p \in R_n$ satisfies the edge condition for the moment graph Γ_G if for all edges $\{\sigma, \tau\}$ of $\Gamma(G)$ with label (i, j), the polynomial

$$p_\sigma(t_1, \ldots, t_n) - p_\tau(t_1, \ldots, t_n)$$

is divisible by $t_i - t_j$.

$H_T^*(\mathcal{H}_G)$ is isomorphic to the subring of R_n whose elements satisfy the edge condition for Γ_G.

Tymoczko's representation

$\mathfrak{S}_n$ acts on $p \in H_T^*(\mathcal{H}_G)$ by

$$(\sigma p)_\tau(t_1, \ldots, t_n) = p_{\sigma^{-1}\tau}(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$$

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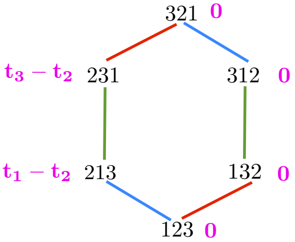

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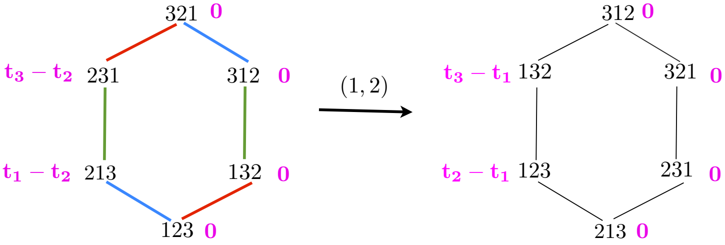

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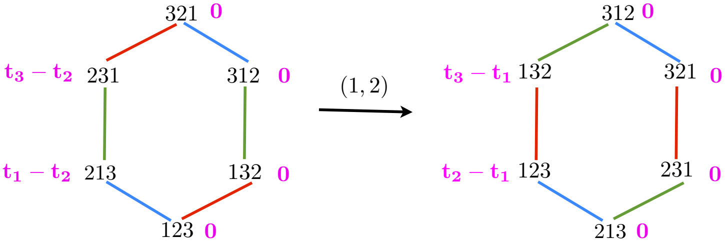

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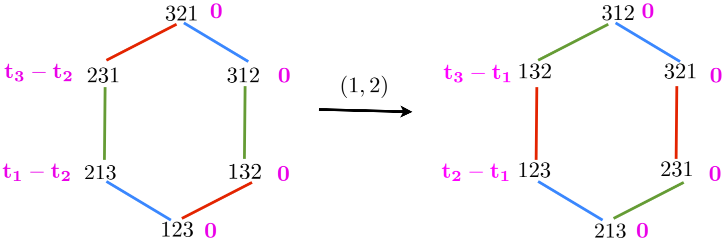


$$H^*(\mathcal{H}_G) \cong H_T^*(\mathcal{H}_G)/(\langle t_1, \ldots, t_n\rangle H_T^*(\mathcal{H}_G))$$

Since $\langle t_1, \ldots, t_n\rangle H_T^*(\mathcal{H}_G)$ is invariant under the action of $\mathfrak{S}_n$, the representation of $\mathfrak{S}_n$ on $H_T^*(\mathcal{H}_G)$ induces a representation on the graded ring $H^*(\mathcal{H}_G)$.

The proofs of $\omega X_G(\mathbf{x}, t) = \sum_{j\geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G)t^j$

- Brosnan and Chow reduce the problem of computing Tymacczko's representation of $\mathfrak{S}_n$ on regular semisimple Hessenberg varieties to that of computing the Betti numbers of regular (but not nec. semisimple) Hessenberg varieties. To do this they use results from the theory of local systems and perverse sheaves. In particular they use the local invariant cycle theorem of Beilinson-Bernstein-Deligne
- Guay-Paquet introduces a new Hopf algebra on labeled graphs to recursivley decompose the regular semisimple Hessenberg varieties.

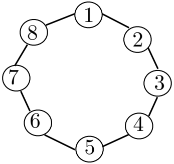
Other recently discovered connections with $X_G(\mathbf{x}, t)$

- Hecke algebra characters evaluated at Kazhdan-Lusztig basis elements: Clearman, Hyatt, Shelton and Skandera (2015). This is a t-analog of work of Haiman (1993).

  The connections with Hecke algebra characters and with Hessenberg varieties yield a connection between Hecke algebra characters and Hessenberg varieties of type A. Schneider and Shareshian have obtained a direct connection and have conjecturally generalized it to other types.

- LLT polynomials and Macdonald polynomials: Haglund and Wilson (2016).

Chromatic quasisymmetric function of the cycle graph

$C_8 =$ 

Not a unit interval graph

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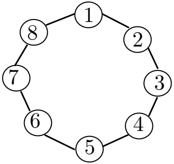
$C_8 =$ 

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Theorem (Stanley (1995))

$$\sum_{n\geq 2} X_{C_n}(\mathbf{x}) z^n = \frac{\sum_{k\geq 2} k(k-1)e_k z^k}{1-\sum_{k\geq 2}(k-1)e_k z^k}.$$

Consequently $X_{C_n}(\mathbf{x})$ is e-positive.

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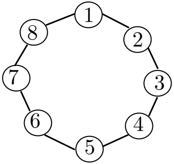
$C_8 =$ 

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Theorem (Ellzey and MW)

$$\sum_{n\geq 2} X_{C_n}(\mathbf{x}, t) z^n = \frac{\sum_{k\geq 2}([2]_t [k]_t + k t^2 [k-3]_t) e_k z^k}{1 - t \sum_{k\geq 2} [k-1]_t e_k z^k}$$

Consequently $X_{C_n}(\mathbf{x}, t)$ is e-positive.

t-analog: $[n]_t := 1 + t + \cdots + t^{n-1}$.

The right definition - suggested by Stanley

original definition:

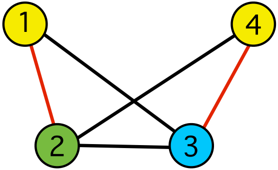
Chromatic quasisymmetric function for labeled graphs

$$X_G(\mathbf{x}, t) := \sum_{c \in C(G)} t^{\mathrm{des}(c)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

where

$$\mathrm{des}(c) := |\{\{i, j\} \in E(G) : i < j \text{ and } c(i) > c(j)\}|.$$

green $<$ yellow $<$ blue



$des(c) = 2$

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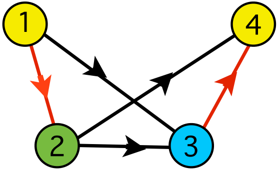


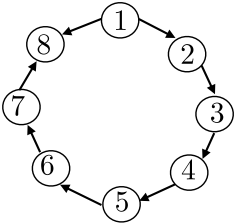
$des(c) = 2$

For a digraph $\overrightarrow{G}$

$$des(c) : |\{(i,j) \in E(\overrightarrow{G}) : c(i) > c(j)\}|$$

labeled graphs $\equiv$ acyclic digraphs

The directed cycle

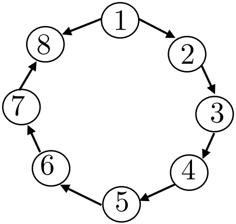


Theorem (Ellzey and MW (2017))

$$\sum_{n\geq 2} X_{C_n}(\mathbf{x},t)z^n = \frac{\sum_{k\geq 2}([2]_t[k]_t + kt^2[k-3]_t)e_k z^k}{1 - t\sum_{k\geq 2}[k-1]_t e_k z^k}$$

Consequently $X_{C_n}(\mathbf{x},t)$ is e-positive.

The directed cycle



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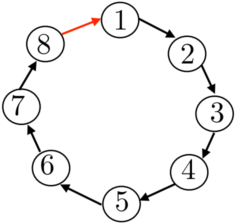

Theorem (Ellzey (2016))

$$\sum_{n\geq 2} X_{\vec{C}_n}(\mathbf{x},t)z^n = \frac{t\sum_{k\geq 2} k[k-1]_t e_k z^k}{1 - t\sum_{k\geq 2}[k-1]_t e_k z^k}.$$

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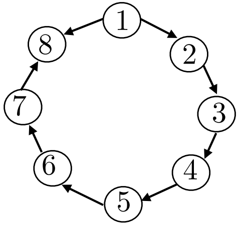

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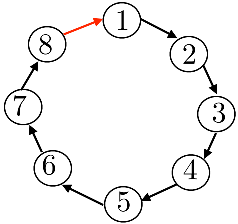

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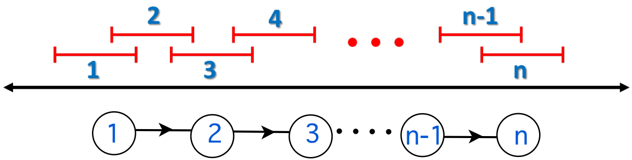
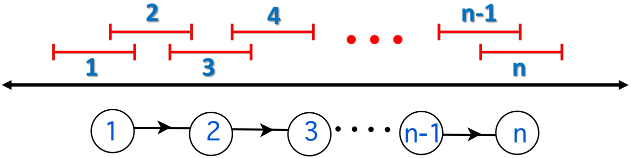
Consequently $X_{\vec{C}_n}(\mathbf{x},t)$ is e-positive.

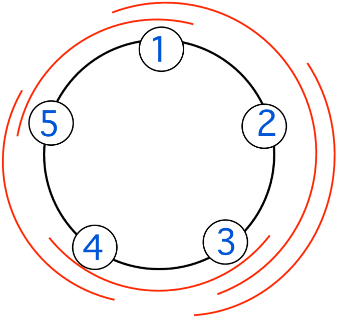
We used the second theorem to prove a result on restricted Smirnov words, which implies both theorems.

Cyclic version of unit interval digraph

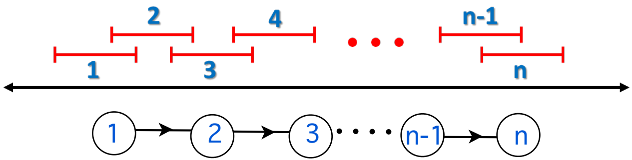


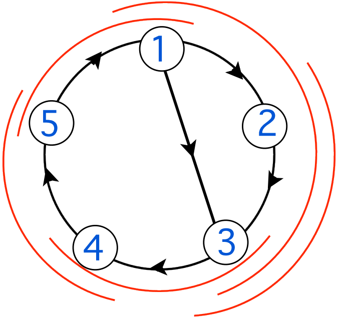
Cyclic version of unit interval digraph





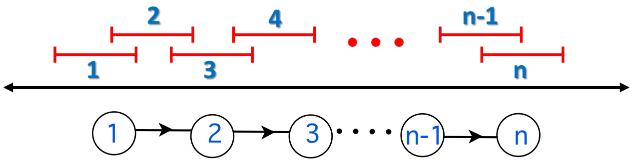
Cyclic version of unit interval digraph

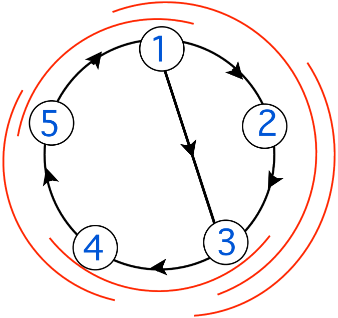




circular indifference digraph

Cyclic version of unit interval digraph





circular indifference digraph

Theorem (Ellzey (2016))

If $\overrightarrow{G}$ is a circular indifference digraph then $\omega X_{\overrightarrow{G}}(\mathbf{x}, t)$ is symmetric and p-positive.

Conjecture

If $\overrightarrow{G}$ is a circular indifference digraph then $X_{\overrightarrow{G}}(\mathbf{x}, t)$ is e-positive and e-unimodal.

Main open problems

- Carry out the remaining step of our approach to proving the Stanley-Stembridge e-positivity conjecture: Prove that Tymoczko's representation of $\mathfrak{S}_n$ on the cohomology of any regular semisimple Hessenberg variety is a permutation representation for which each point stabilizer is a Young subgroup.

- Generalize this approach to circular indifference digraphs: Find a geometric interpretation of the extension of chromatic quasisymmetric functions to circular indifference digraphs and use it to obtain e-positivity.