Combinatorial positive valuations

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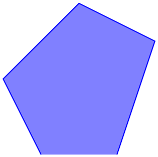
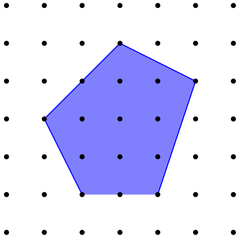
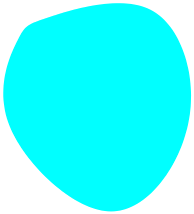
Volume vs. discrete volume

Polytopes and convex bodies







Λ	:	$\mathbb{R}^d$ or $\mathbb{Z}^d$
$\mathcal{P}(\Lambda)$	:	set of all polytopes with vertices in Λ, called Λ-**polytopes**
$\mathcal{K}$	:	set of all convex bodies in $\mathbb{R}^d$

Volume

$\mathrm{V}_d(P) = \int_P 1 d\mu$ d-dimensional **volume** of $P \in \mathcal{K}$

Properties:

- homogeneous: $\mathrm{V}_d(\lambda P) = \lambda^d \mathrm{V}_d(P)$ for all $\lambda \geq 0$
- monotone: $\mathrm{V}_d(P) \leq \mathrm{V}_d(Q)$ whenever $P \subseteq Q$
- (continuous)
- rigid-motion invariant
- **valuation** property:
  1. $\mathrm{V}_d(\emptyset) = 0$,
  2. for $P, P' \in \mathcal{K}$ such that $P \cup P' \in \mathcal{K}$
$$\mathrm{V}_d(P \cup P') = \mathrm{V}_d(P) + \mathrm{V}_d(P') - \mathrm{V}_d(P \cap P').$$

Hadwiger's Characterization Theorem

Theorem (Hadwiger '57)

The family of continuous, real-valued, rigid-motion invariant valuations on convex bodies is a $(d+1)$*-dimensional vector space spanned by the quermassintegrals* $W_0, W_1, \ldots, W_d$*.*

Hadwiger's Characterization Theorem

Minkowski sum: For $P, Q \in \mathcal{K}$

$$P + Q \ := \ \{p + q \colon p \in P, q \in Q\}$$

Steiner polynomial:

$$\mathrm{V}_d(P + n\mathcal{B}_d) \ = \ \sum_{i=0}^{d} \binom{d}{i} W_i(P) n^i$$

where $\mathcal{B}_d$ is the unit ball and $W_i(P)$ is the i-th quermassintegral.

Properties:

- valuation
- rigid-motion invariant
- monotone

Hadwiger's Characterization Theorem

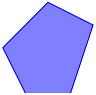
Theorem (Hadwiger '57)

A continuous, rigid-motion invariant valuation $\varphi\colon \mathcal{K} \to \mathbb{R}$ *is positive or monotone if and only if there are* $c_0, c_1, \ldots, c_d \geq 0$ *with*

$$\varphi = c_0 W_0 + c_1 W_1 + \cdots + c_d W_d.$$

Discrete volume

$\mathrm{E}(P) = \left|P \cap \mathbb{Z}^d\right|$ **discrete volume** of a lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$



volume	**discrete volume**
valuation	
monotone	
rigid-motion invariant	
homogeneous	

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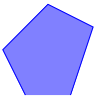

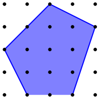

volume
valuation
monotone
rigid-motion invariant
homogeneous

discrete volume
valuation

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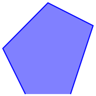
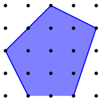


volume
valuation
monotone
rigid-motion invariant
homogeneous



discrete volume
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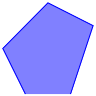

volume
valuation
monotone
rigid-motion invariant
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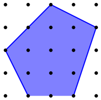

discrete volume
valuation
monotone
lattice invariant

A valuation $\varphi\colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ is **lattice invariant** if for all $T \in \mathrm{GL}_d(\mathbb{Z}^d)$ and $t \in \mathbb{Z}^d$

$$\varphi(T(P) + t) = \varphi(P).$$

Discrete volume

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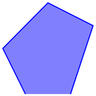

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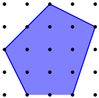

discrete volume

valuation
monotone
lattice invariant
polynomial

Polynomiality

Theorem (Ehrhart'62)

For every lattice polytope P in $\mathbb{R}^d$

$$n \mapsto |nP \cap \mathbb{Z}^d|$$

agrees with a polynomial $\mathrm{E}_P(n)$ of degree $\dim P$ *for integers $n \geq 1$.*

$\mathrm{E}_P(n)$ is called the **Ehrhart polynomial** of P.

Central Questions

- Which polynomials are Ehrhart polynomials?
- Interpretation of coefficients

Betke-Kneser Theorem

Theorem (Betke-Kneser '85)

The family of lattice invariant valuations form a $(d+1)$-dimensional vectorspace spanned by the coefficients $E_0, E_1, \ldots, E_d$ of the Ehrhart polynomial.

Betke-Kneser Theorem

Ehrhart polynomial:

$$\mathrm{E}_P(n) = \mathrm{E}_0(P) + \mathrm{E}_1(P)n + \cdots + \mathrm{E}_d(P)n^d$$

Each coefficient $\mathrm{E}_i \colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ is a...

- valuation
- lattice invariant
- in general **not monotone/positive**

Question: Is there a classification for positive/monotone lattice invariant valuations?

Combinatorial positive valuations

Ehrhart series and h^*-polynomial

Ehrhart series

The **Ehrhart series** of an r-dimensional lattice polytope $P \subset \mathbb{R}^d$ is defined by

$$\sum_{n \geq 0} \mathrm{E}_P(n) t^n \quad = \quad \frac{h_0^*(P) + h_1^*(P)t + \cdots + h_r^*(P)t^r}{(1-t)^{r+1}} .$$

The numerator polynomial $h^*(P)(t) = \sum_{i=0}^{r} h_i^*(P)t^i$ is called the h^*-**polynomial** of P. The vector $(h_0^*(P), h_1^*(P), \ldots, h_d^*(P))$ is called the h^*-**vector** of P, where $h_i^*(P) := 0$ for $i > r$.

Ehrhart series and h^*-polynomial

Ehrhart series

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h^*-polynomial and coefficients of $\mathrm{E}_P(n)$

Expansion into a binomial basis:

$$\mathrm{E}_P(n) = h_0^*(P)\binom{n+r}{r} + h_1^*(P)\binom{n+r-1}{r} + \cdots + h_r^*(P)\binom{n}{r} \,.$$

Stanley's Nonnegativity Theorem

Theorem (Stanley '80, '93)

Let $P, Q \in \mathcal{P}(\mathbb{Z}^d)$ be lattice polytopes. Then

$$h_i^*(P) \geq 0 \qquad \textbf{(positivity)}$$

for all $0 \leq i \leq d$ and, if $P \subseteq Q$, then

$$h_i^*(P) \leq h_i^*(Q) \qquad \textbf{(monotonicity)}\ .$$

Translation-invariant valuations

A map $\varphi\colon \mathcal{P}(\Lambda) \to \mathbb{R}$ is a **translation-invariant valuation** if

- $\varphi(\emptyset) = 0$,
- for $P, P' \in \mathcal{P}(\Lambda)$ such that $P \cup P', P \cap P' \in \mathcal{P}(\Lambda)$
$$\varphi(P \cup P') = \varphi(P) + \varphi(P') - \varphi(P \cap P'),$$
- for all $P \in \mathcal{P}(\Lambda)$ and all $t \in \Lambda$
$$\varphi(P + t) = \varphi(P).$$

Examples:
$\Lambda = \mathbb{R}^d$: Vol, W_i, χ
$\Lambda = \mathbb{Z}^d$: $|P \cap \mathbb{Z}^d|$, solid-angles

Theorem (McMullen '77)

Let φ be a translation-invariant valuation and P be a Λ-polytope. Then $\varphi(nP)$ agrees with a polynomial $\varphi_P(n)$ of degree at most $\dim(P)$ for integers $n \geq 0$.

Combinatorial positivity

For an r-dimensional polytope $P \in \mathcal{P}(\Lambda)$ and a translation-invariant valuation φ let

$$\sum_{n \geq 0} \varphi_P(n) t^n = \frac{h_0^\varphi(P) + h_1^\varphi(P) t + \cdots + h_r^\varphi(P) t^r}{(1-t)^{r+1}} \, .$$

$h^\varphi(P) = (h_0^\varphi(P), \ldots, h_d^\varphi(P))$: h^*-**vector** of P with respect to φ, where $h_i^\varphi(P) := 0$ for $i > r$.

We define

φ **combinatorially positive** : $0 \leq h_i^\varphi(P)$ $\quad \forall P \in \mathcal{P}(\Lambda), \forall i.$

φ **combinatorially monotone** : $h_i^\varphi(P) \leq h_i^\varphi(Q)$ $\quad \forall P \subseteq Q \in \mathcal{P}(\Lambda), \forall i.$

Examples

Combinatorially positive:

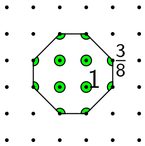
- $\mathrm{E}(P) = \left|P \cap \mathbb{Z}^d\right|$ discrete volume (Stanley '80)).
- $\mathrm{V}_d(P)$: If $P \subset \mathbb{R}^d$ is a d-dimensional polytope, then

$$\sum_{n \geq 0} \mathrm{V}_d(nP) t^n = \mathrm{V}_d(P) \sum_{n \geq 0} n^d t^n = \mathrm{V}_d(P) \frac{A_d(t)}{(1-t)^{d+1}}$$

where $A_d(t)$ is the *Eulerian polynomial*.

- Solid-angle sum (Beck, Robins, Sam '10): $A(P) = \sum_{x \in \mathbb{Z}^d} \omega_P(x)$, where

$\omega_P(x) = \lim_{r \to 0} \frac{\mathrm{Vol}_d(B(r,x) \cap P)}{\mathrm{Vol}_d(B(r,x))}$



Not combinatorially positive: $\chi(P)$ Euler characteristic

$$\sum_{n \geq 0} \chi(nP) t^n = \frac{1}{1-t} = \frac{(1-t)^{\dim P}}{(1-t)^{\dim P + 1}}$$

Combinatorial positivity

We define

$$\varphi(\mathrm{relint}(P)) := \sum_{F \subseteq P \text{ face}} (-1)^{\dim P - \dim F} \varphi(F) \,.$$

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Theorem (J., Sanyal '15)

Let $\varphi\colon \mathcal{P}(\Lambda) \to \mathbb{R}$ a translation-invariant valuation. Then the following are equivalent:

(i) *φ is combinatorially positive.*

(ii) *φ is combinatorially monotone.*

(iii) *$\varphi(\mathrm{relint}(\Delta)) \geq 0$ for all simplices $\Delta \in \mathcal{P}(\Lambda)$.*

A volume characterization theorem

Theorem (J., Sanyal '15)

Let $\varphi\colon \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ *be a combinatorial positive valuation. Then*

$$\varphi = \lambda \cdot \mathsf{V}_d$$

for some $\lambda \geq 0$.

Discrete Hadwiger-type theorem

$$\mathrm{E}_P(n) \; = \; f_0^*(P)\binom{n-1}{0} + f_1^*(P)\binom{n-1}{1} + \cdots + f_d^*(P)\binom{n-1}{d}$$

Properties:
The coefficients f_i^* are

- lattice-invariant valuations
- nonnegative (Breuer '12)
- combinatorially positive (J. , Sanyal '15)

Discrete Hadwiger-type theorem

Theorem (J., Sanyal '15)

A lattice-invariant valuation $\varphi\colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ *is combinatorially positive if and only if there are* $c_0, c_1, \ldots, c_d \geq 0$ *such that*

$$\varphi \;=\; c_0 f_0^* + c_1 f_1^* + \cdots + c_d f_d^*$$

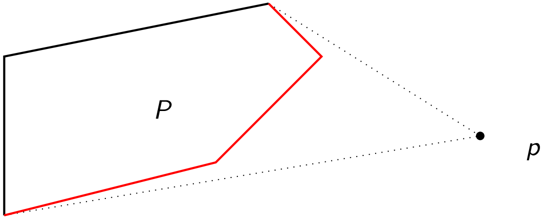
Half-open polytopes

Half-open polytopes

A face F of a polytope P is **visible** from some $p \in \mathbb{R}^d$ if for all $q \in F$

$$(q, p] \cap P = \emptyset.$$

Let $H_p(P)$ denote the **half-open polytope** without the visible faces.



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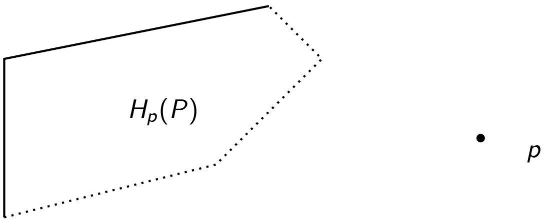

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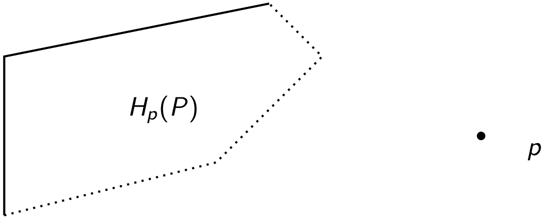


If $F_1, \ldots, F_m$ are the visible facets of P then

$$\varphi_{H_p P}(n) := \varphi(nH_p P) = \varphi(nP) - \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \varphi(n \bigcap_{j \in I} F_j)$$

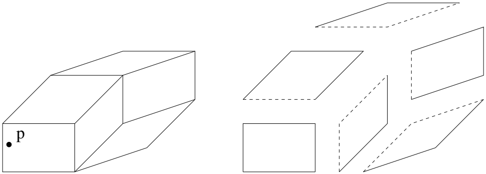
is a polynomial.

Half-open decomposition

Theorem (Köppe, Verdoolaege '08)

If $P = P_1 \cup \cdots \cup P_m$ is a dissection, and $p \in P$ generic, then

$$P = \biguplus_{i=1}^{m} H_p P_i.$$



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In particular,

$$\varphi_P(n) = \sum_{i=1}^{m} \varphi_{H_p P_i}(n),$$

and

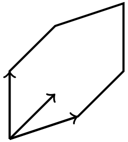
$$h^{\varphi}(P)(t) = \sum_{i=1}^{m} h^{\varphi}(H_p P_i)(t).$$

h^*-polynomials of zonotopes

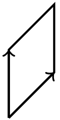
Lattice parallelepipeds and lattice zonotopes

Lattice zonotope: $v_1, \ldots, v_k \in \mathbb{Z}^d$

$$\mathcal{Z}(v_1, \ldots, v_k) = \left\{ \sum_{i \in [k]} \lambda_i v_i : 0 \leq \lambda_i \leq 1 \right\}$$



Lattice parallelepiped: $v_1, \ldots, v_k \in \mathbb{Z}^d$ linearly independent



Ehrhart polynomial of zonotopes

Theorem (Stanley '91)

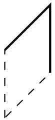
Let $\mathcal{Z}(v_1, \ldots, v_k)$ be a lattice zonotope generated by the set of vectors $V = \{v_1, \ldots, v_k\} \subset \mathbb{Z}^d$. Then

$$\mathrm{E}_{\mathcal{Z}(v_1,\ldots,v_k)}(n) \ = \ \sum_I g(I)\, n^{|I|}$$

where I ranges over all linearly independent subsets of V, and $g(I)$ denotes the greatest common divisor of all maximal minors of the matrix with column vectors I.

Geometric meaning of the coefficients:

$$g(I) = |\Pi(I) \cap \mathbb{Z}^d| \quad \text{where} \quad \Pi(I) = \left\{ \sum_{i \in [I]} \lambda_i v_i : 0 < \lambda_i \leq 1 \right\}$$



Integer decomposition property

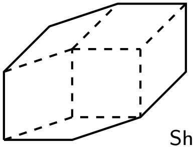
A lattice polytope $P \subset \mathbb{R}^d$ has the **integer decomposition property (IDP)** if for all integers $n \geq 1$ and all $p \in nP \cap \mathbb{Z}^d$

$$p = p_1 + \cdots + p_n$$

for some $p_1, \ldots, p_n \in P \cap \mathbb{Z}^d$.

Examples

- unimodular simplex
- lattice parallelepiped
- lattice zonotope



Shephard '74

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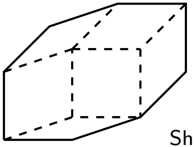


Shephard '74

Conjecture (Stanley '89; Hibi, Ohsugi '06; Schepers, Van Langenhoven '13)

If P is IDP then the h^*-polynomial of P has unimodal coefficients.

unimodal: $h_0^* \leq h_1^* \leq \cdots \leq h_k^* \geq \cdots \geq h_d^*$ for some k

Lattice zonotopes

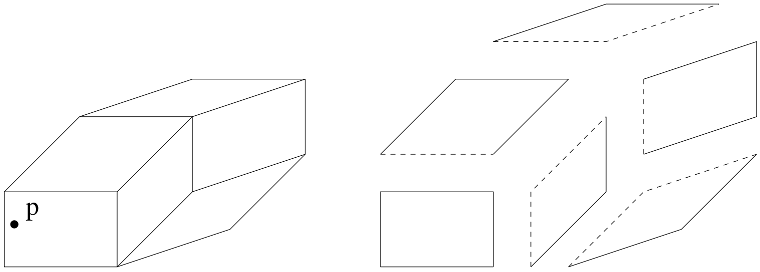
Theorem (Schepers, Van Langenhoven '13)

The h^*-polynomial of any lattice parallelepiped is unimodal.

Theorem (Beck, J., McCullough '17)

The h^*-polynomial of any lattice zonotope is real-rooted.

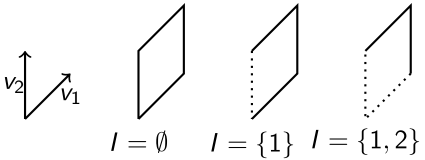
Decomposing



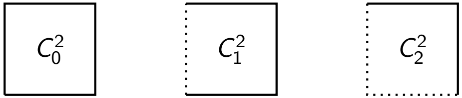
Half-open parallelepipeds

For $v_1, \ldots, v_d \in \mathbb{Z}^d$ linear independent and $I \subseteq [d]$

$$\text{⬗}(I)(v_1, \ldots, v_d) := \left\{ \sum_{i \in [d]} \lambda_i v_i : 0 \le \lambda_i \le 1, 0 < \lambda_i \text{ if } i \in I \right\}$$



Special case: **Half-open unit cubes** $C_j^d = [0,1]^d \setminus \{x_1 = 0, \ldots, x_j = 0\}$



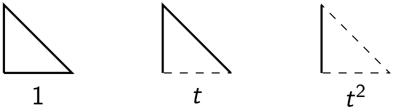
Half-open unimodular simplices

Half-open unimodular simplices
For a unimodular d-simplex Δ with facets $F_1, \ldots, F_{d+1}$

$$\mathrm{E}_{\Delta}(n) \;=\; \binom{n+d}{d} \;\Rightarrow\; h^*(\Delta)(t) \;=\; 1$$

More generally, for $0 \leq i \leq d$

$$\mathrm{E}_{\Delta \setminus \bigcup_{k=1}^{i} F_k}(n) \;=\; \binom{n+d-i}{d} \;\Rightarrow\; h^*(\Delta)(t) \;=\; t^i$$



Eulerian numbers

We call $i \in [d-1]$ a **descent** of a permutation $\sigma \in S_d$ if $\sigma(i) > \sigma(i+1)$. The number of descents of σ is denoted by $\operatorname{des}\sigma$ and

$$A(d,t) \ = \ \sum_{\sigma \in S_d} t^{\operatorname{des}\sigma}$$

is the **Eulerian polynomial**.

Example:

$$\begin{pmatrix} 1 & \color{red}{2} & 3 \\ 2 & \color{red}{3} & 1 \end{pmatrix} \in S_3$$

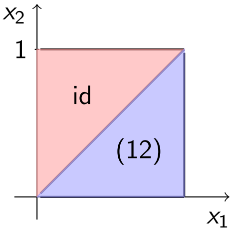
$$123 \quad 1\color{red}{3}2 \quad \color{red}{2}13 \quad 2\color{red}{3}1 \quad \color{red}{3}12 \quad \color{red}{32}1$$

$$A(3,t) \ = \ 1 + 4t + t^2$$

Unit cubes

Partition of unit cube $C^d = [0,1]^d$

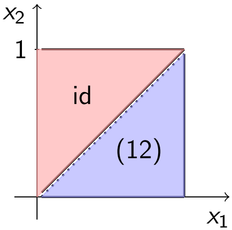
$$C^d \;=\; \bigcup_{\sigma \in S_d} \{\mathbf{x} \in C^d \colon x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(d)}\}$$



Unit cubes

Partition of unit cube $C^d = [0,1]^d$

$$C^d \;=\; \biguplus_{\sigma \in S_d} \{\mathbf{x} \in C^d \colon x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(d)},$$
$$x_{\sigma(i)} < x_{\sigma(i+1)}, \text{ if } i \text{ descent of } \sigma\}$$



Unit cubes

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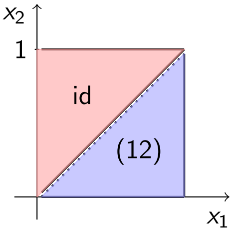
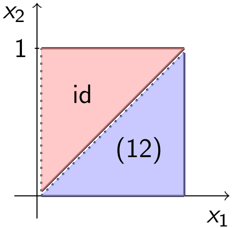


$$h^*(C^d)(t) = \sum_{\sigma \in S_d} t^{des \sigma} = A(d,t)$$

Half-open unit cubes

Partition of half-open unit cube $C_j^d = [0,1]^d \setminus \{x_1 = 0, \ldots, x_j = 0\}$

$$C_j^d = \biguplus_{\sigma \in S_d} \{\mathbf{x} \in C_j^d : x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(d)},$$
$$x_{\sigma(i)} < x_{\sigma(i+1)}, \text{ if } i \text{ descent of } \sigma\}$$



$$h^*(C_j^d)(t) = \sum_{\sigma \in S_d} t^{\mathrm{des}_j \sigma} \quad \text{where} \quad \mathrm{des}_j \sigma = \begin{cases} \mathrm{des}\,\sigma + 1 & \text{if } \sigma(1) \le j, \\ \mathrm{des}\,\sigma & \text{otherwise.} \end{cases}$$

Refined Eulerian numbers

$$\{\sigma \in S_d : \mathrm{des}_j \sigma = k\} \cong \{\sigma \in S_{d+1} : \mathrm{des}\, \sigma = k, \sigma(1) = j+1\}$$

Example: $d = 5$, $j = 3$

$$24351 \;\mapsto\; 424351 \mapsto 425361$$

The j-**Eulerian polynomial** is

$$A_j(d,k) \;=\; \sum_{\substack{\sigma \in S_d \\ \sigma(1)=j}} t^{\mathrm{des}\,\sigma}$$

Theorem (Beck, J., McCullough '17)

$$h^*(C_j^d)(t) = A_{j+1}(d+1,t).$$

h^*-polynomials of half-open parallelepipeds

Theorem (Beck, J., McCullough '17)

$$h^*\left(\lozenge(I)\right)(t) = \sum_{K \subseteq [d]} |\square(K) \cap \mathbb{Z}^d| A_{|I \cup K|+1}(d+1, t)\,,$$

where

$$\square(K) := \left\{ \sum_{i \in [K]} \lambda_i v_i : 0 < \lambda_i < 1 \right\}\,.$$

h^*-polynomials of half-open parallelepipeds

Theorem (Beck, J., McCullough '17)

Let φ be a translation-invariant valuation. Then

$$h^{\varphi}\left(\lozenge(I)\right)(t) = \sum_{K \subseteq [d]} \varphi(\square(K)) A_{|I \cup K|+1}(d+1, t)\,,$$

where

$$\square(K) := \left\{ \sum_{i \in [K]} \lambda_i v_i : 0 < \lambda_i < 1 \right\}.$$

Compatible refined Eulerian polynomials

Theorem (Savage, Visontai '15)

The j-Eulerian polynomials are **compatible**, *i.e. for all $c_1, \ldots, c_d \geq 0$*

$$c_1 A_1(d,t) + c_2 A_2(d,t) + \cdots + c_d A_d(d,t)$$

is real-rooted.

Theorem (Beck, J., McCullough '17)

Let $\mathcal{Z}$ be a lattice zonotope and let φ be a combinatorial positive valuation. Then $h^{\varphi}(\mathcal{Z})(t)$ is real-rooted.

The cone of h^*-polynomials

Theorem (Beck, J., McCullough '17)

Let $d \geq 1$. Then the convex hull of the set of all h^*-polynomials of lattice zonotopes/parallelepipeds equals

$$A_1(d+1,t) + \mathbb{R}_{\geq 0} A_2(d+1,t) + \cdots + \mathbb{R}_{\geq 0} A_{d+1}(d+1,t)\,.$$

Ehrhart tensor polynomials

Discrete moment tensors

Discrete moment tensor of rank r (Böröczky, Ludwig '17):

$$L^r(P) := \sum_{x \in P \cap \mathbb{Z}^d} x^r\,,$$

where $x^r = \underbrace{x \otimes \cdots \otimes x}_{r} \in \mathbb{T}^r$ (symmetric tensors).

Translation-covariance:

$$L^r(P+t) := \sum_{i=0}^{r} \binom{r}{i} L^{r-i}(P) t^r$$

Ehrhart tensor polynomials

Theorem (*)

There exist $L_i^r(P) : \mathcal{P}(\mathbb{Z}^d) \to \mathbb{T}^r$ *for all* $1 \leq i \leq d + r$ *such that*

$$L^r(mP) = \sum_{i=0}^{d+r} L_i^r(P) m^i$$

for any $m \in \mathbb{Z} \geq 0$ *and* $P \in \mathcal{P}(\mathbb{Z}^d)$. $L^r(mP)$ *is called the* **Ehrhart tensor polynomial** *of rank* r.

* Ehrhart (1962), $r = 0$
* McMullen (1977), $r = 1$
* Khovanskiĭ and Pukhlikov (1992)
* Ludwig, Silverstein (2017+), $r \geq 0$

h^r-tensor polynomial

For a d-dimensional lattice polytope P

$$L^r(nP) \;=\; h_0^r(P)\binom{n+d+r}{d+r} + h_1^r(P)\binom{n+d+r-1}{d+r} + \cdots + h_{d+r}^r(P)\binom{n}{d+r}$$

$(h_0^r(P), h_1^r(P), \ldots, h_{d+r}^r(P))$ is the h^r-**tensor polynomial** of P, and its entries are the h^r-**tensors**.

Question

Are the h^r-tensors positive/monotone for all lattice polytopes P?

- depends on order on $\mathbb{T}^r$
- canonical choice: positive semidefinite matrices for $r = 2$

Positivity of h^r-tensors

- h^2-tensors can be negative on half-open polytopes
- h^2-tensors in general not monotone

Theorem (Berg, J., Silverstein 17+)

For lattice polygons, all h^2-tensors are positive semidefinite.

Conjecture (Berg, J., Silverstein 17+)

The h^2-tensors of any lattice polytope are positive semidefinite.

The End

- Matthias Beck, Katharina Jochemko, Emily McCullough: *h*-polynomials of zonotopes, Transactions of the AMS, in press*
- Sören Berg, Katharina Jochemko, Laura Silverstein: *Ehrhart tensor polynomials, arXiv:1706.01738*
- Katharina Jochemko, Raman Sanyal: *Combinatorial positivity of translation-invariant valuations and a discrete Hadwiger theorem, Journ. Eur. Math. Soc. (JEMS), in press*.

Thank you!