Symmetric Sums of Squares

Annie Raymond
University of Washington → MSRI → University of Massachusetts

Joint with James Saunderson (Monash University),
Mohit Singh (Georgia Tech), and Rekha Thomas (UW)

October 10, 2017

Goal

Certify the nonnegativity of a symmetric polynomial over the hypercube.

Our key result: the runtime does not depend on the number of variables of the polynomial

1. Background
2. Our setting
3. Results
4. Flag algebras
5. Future work

Finding sos certificates

- $p \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \ldots, x_n]$ such that $\deg(p) = 2d$
- $[x]_d := (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^d)^\top$
  $= \text{vector of monomials in } \mathbb{R}[\mathbf{x}] \text{ of degree} \leq d$
- p sos $\Leftrightarrow\ \exists\ Q \succeq 0$ such that $p = [x]_d^\top Q[x]_d$

Finding sos certificates

- $p \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \ldots, x_n]$ such that $\deg(p) = 2d$
- $[x]_d := (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^d)^\top$
  $= \text{vector of monomials in } \mathbb{R}[\mathbf{x}] \text{ of degree} \leq d$
- p sos $\Leftrightarrow \ \exists\ Q \succeq 0$ such that $p = [x]_d^\top Q [x]_d$
  $= [x]_d^\top B B^\top [x]_d = ([x]_d^\top B)([x]_d^\top B)^\top$

Finding sos certificates

- $p \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \ldots, x_n]$ such that $\deg(p) = 2d$
- $[x]_d := (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^d)^\top$
  $= \text{vector of monomials in } \mathbb{R}[\mathbf{x}] \text{ of degree } \leq d$
- p sos $\Leftrightarrow\ \exists\ Q \succeq 0$ such that $p = [x]_d^\top Q [x]_d$
  $= [x]_d^\top BB^\top [x]_d = ([x]_d^\top B)([x]_d^\top B)^\top$

Example

$$p = x_1^2 - x_1x_2 + x_2^2 + 1 = \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}$$

$$= 1 + \frac{3}{4}(x_1 - x_2)^2 + \frac{1}{4}(x_1 + x_2)^2$$

Sums of squares modulo an ideal

Goal

Certify $p \geq 0$ over the solutions of a system of polynomial equations.

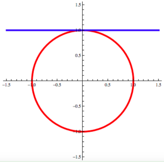
Sums of squares modulo an ideal

Goal

Certify $p \geq 0$ over the solutions of a system of polynomial equations.

Example

Show that $1 - y \geq 0$ whenever $x^2 + y^2 = 1$



$$1 - y = \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y-1}{\sqrt{2}}\right)^2 - \frac{1}{2}(x^2 + y^2 - 1)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} - \frac{1}{2}(x^2 + y^2 - 1)$$

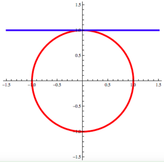
Sums of squares modulo an ideal

Goal

Certify $p \geq 0$ over the solutions of a system of polynomial equations.

Example

Show that $1 - y \geq 0$ whenever $x^2 + y^2 = 1$



$$1 - y = \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y-1}{\sqrt{2}}\right)^2 - \frac{1}{2}(x^2 + y^2 - 1)$$

$$= \frac{1}{2}\begin{pmatrix}1 & x & y\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ x\\ y\end{pmatrix} - \frac{1}{2}(x^2 + y^2 - 1)$$

- Ideal $\mathcal{I} \subseteq \mathbb{R}[\mathbf{x}]$
- $V_{\mathbb{R}}(\mathcal{I})$=its real variety
- p is sos modulo $\mathcal{I}$ if $p \equiv \sum_{i=1}^{l} f_i^2 \mod \mathcal{I}$
  (i.e., if $\exists\ h \in \mathcal{I}$ such that $p = \sum_{i=1}^{l} f_i^2 + h$)
- p is d-sos mod $\mathcal{I}$ if $p \equiv \sum_{i=1}^{l} f_i^2 \mod \mathcal{I}$ where $\deg(f_i) \leq d\ \forall\ i$

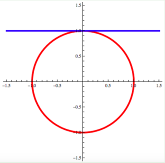
Sums of squares modulo an ideal

Goal

Certify $p \geq 0$ over the solutions of a system of polynomial equations.

Example

Show that $1 - y \geq 0$ whenever $x^2 + y^2 = 1$



$$1 - y = \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y-1}{\sqrt{2}}\right)^2 - \frac{1}{2}(x^2 + y^2 - 1)$$

$$= \frac{1}{2}\begin{pmatrix}1 & x & y\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ x\\ y\end{pmatrix} - \frac{1}{2}(x^2 + y^2 - 1)$$

- Ideal $\mathcal{I} \subseteq \mathbb{R}[\mathbf{x}]$
- $V_{\mathbb{R}}(\mathcal{I})$=its real variety
- p is sos modulo $\mathcal{I}$ if $p \equiv \sum_{i=1}^{l} f_i^2 \mod \mathcal{I}$
  (i.e., if $\exists\ h \in \mathcal{I}$ such that $p = \sum_{i=1}^{l} f_i^2 + h$)
- p is d-sos mod $\mathcal{I}$ if $p \equiv \sum_{i=1}^{l} f_i^2 \mod \mathcal{I}$ where $\deg(f_i) \leq d\ \forall\ i \Leftrightarrow\ \exists\ Q \succeq 0$ such that $p \equiv v^\top Q v \mod \mathcal{I}$ (semidefinite programming can find Q in $n^{O(d)}$-time)

Our problem

Let $\mathcal{V}_{n,k} = \{0,1\}^{\binom{n}{k}}$ be the k-subset discrete hypercube
$\rightarrow$ coordinates indexed by k-element subsets of $[n]$

Goal

Minimize a symmetric* polynomial over $\mathcal{V}_{n,k}$

*symmetric = $\mathfrak{S}_n$-invariant

$$\mathfrak{s} \cdot x_{i_1 i_2 \ldots i_k} = x_{\mathfrak{s}(i_1)\mathfrak{s}(i_2)\ldots \mathrm{s}(i_k)} \; \forall \mathfrak{s} \in \mathfrak{S}_n$$

Our problem

Let $\mathcal{V}_{n,k} = \{0,1\}^{\binom{n}{k}}$ be the k-subset discrete hypercube
$\rightarrow$ coordinates indexed by k-element subsets of $[n]$

Goal

Minimize a symmetric* polynomial over $\mathcal{V}_{n,k}$

*symmetric = $\mathfrak{S}_n$-invariant

$$\mathfrak{s} \cdot x_{i_1 i_2 \ldots i_k} = x_{\mathfrak{s}(i_1)\mathfrak{s}(i_2)\ldots \mathrm{s}(i_k)} \ \forall \mathfrak{s} \in \mathfrak{S}_n$$

How?
By finding sos certificates over $\mathcal{V}_{n,k}$ that exploit symmetry, i.e., that we can find in a runtime independent of n.

$k = 1$: see Blekherman, Gouveia, Pfeiffer (2014)
$k \geq 2$: ?

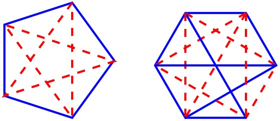
Examples of such problems

- **Turán-type problem**
  Given a fixed graph H, determine the limiting edge density of a H-free graph on n vertices as $n \to \infty$

Examples of such problems

- **Turán-type problem**
  Given a fixed graph H, determine the limiting edge density of a H-free graph on n vertices as $n \to \infty$

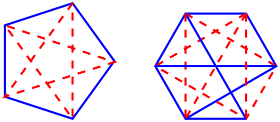
- **Ramsey-type problem**
  Color the edges of K_n ruby or sapphire. Find the smallest n for which you are guaranteed a ruby clique of size r or a sapphire clique of size s



Examples of such problems

- **Turán-type problem**
  Given a fixed graph H, determine the limiting edge density of a H-free graph on n vertices as $n \to \infty$

- **Ramsey-type problem**
  Color the edges of K_n ruby or sapphire. Find the smallest n for which you are guaranteed a ruby clique of size r or a sapphire clique of size s



Focus on $\mathcal{V}_n := \mathcal{V}_{n,2} = \{0,1\}^{\binom{n}{2}}$
$\to$ coordinates are indexed by pairs ij, $1 \leq i < j \leq n$

Passing to optimization - Turán-type problem

Example

Forbidding triangles in a graph on n vertices, find

$$\begin{array}{|lll|}\hline \max & \frac{1}{\binom{n}{2}} \sum_{1\le i<j\le n} x_{ij} & \\ \text{s.t.} & x_{ij}^2 = x_{ij} & \forall 1 \le i < j \le n \\ & x_{ij}x_{jk}x_{ik} = 0 & \forall 1 \le i < j < k \le n \\ \hline \end{array}$$

In particular, show that this is at most $\frac{1}{2} + O(\frac{1}{n})$

$\rightarrow$ show that $\frac{1}{2} + O(\frac{1}{n}) - \frac{1}{\binom{n}{2}} \sum_{1\le i<j\le n} x_{ij} \ge 0$

Issue with passing to optimization - Turán-type problem

Example (continued)

Find $Q \succeq 0$ and $d \in \mathbb{Z}^+$ such that

$$\frac{1}{2} + O\left(\frac{1}{n}\right) - \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} x_{ij} \equiv v^\top Q v \mod \mathcal{I}$$

where

$$\mathcal{I} = \langle x_{ij}^2 - x_{ij} \ \forall 1 \le i < j \le n, \\ x_{ij}x_{jk}x_{ik} \ \forall 1 \le i < j < k \le n \rangle$$

Issue with passing to optimization - Turán-type problem

Example (continued)

Find $Q \succeq 0$ and $d \in \mathbb{Z}^+$ such that

$$\frac{1}{2} + O\left(\frac{1}{n}\right) - \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} x_{ij} \equiv v^\top Q v \mod \mathcal{I}$$

where

$$\mathcal{I} = \langle x_{ij}^2 - x_{ij} \ \forall 1 \leq i < j \leq n, \\ x_{ij} x_{jk} x_{ik} \ \forall 1 \leq i < j < k \leq n \rangle$$

Can we do this with semidefinite programming?
The runtime would be $\binom{n}{2}^{O(d)}$

Issue with passing to optimization - Turán-type problem

Example (continued)

Find $Q \succeq 0$ and $d \in \mathbb{Z}^+$ such that

$$\frac{1}{2} + O\left(\frac{1}{n}\right) - \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} x_{ij} \equiv v^\top Q v \mod \mathcal{I}$$

where

$$\mathcal{I} = \langle x_{ij}^2 - x_{ij} \ \forall 1 \leq i < j \leq n, \\ x_{ij}x_{jk}x_{ik} \ \forall 1 \leq i < j < k \leq n \rangle$$

Can we do this with semidefinite programming?
The runtime would be $\binom{n}{2}^{O(d)} \rightarrow \infty$ as $n \rightarrow \infty$.

Foreshadowing

Example

The following is a sos proof of Mantel's theorem

$$\begin{pmatrix} 1 & q_1 \end{pmatrix} \begin{pmatrix} \frac{(n-1)^2}{2} & -\frac{2(n-1)}{n} \\ -\frac{2(n-1)}{n} & \frac{8}{n^2} \end{pmatrix} \begin{pmatrix} 1 \\ q_1 \end{pmatrix} + \text{sym}\left(\left(q_2\right) \left(\tfrac{8}{n^2}\right) \left(q_2\right) \right)$$

where $q_1 = \sum_{i<j} x_{ij}$ and $q_2 = \sum_{i<j} x_{ij} - \frac{n-2}{2} \sum_{i=1}^{n-1} x_{in}$

Key features of desired sos certificates:

- exploits symmetry
- constant size
- entries are functions of n

Representation theory needed for exploiting symmetry

- $(\mathbb{R}[x]/\mathcal{I})_d =: V = \bigoplus_{\lambda \vdash n} V_\lambda$ isotypic decomposition
  - partition $\lambda = (5, 3, 3, 1)$ for $n = 12$

Representation theory needed for exploiting symmetry

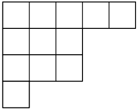
- $(\mathbb{R}[x]/\mathcal{I})_d =: V = \bigoplus_{\lambda \vdash n} V_\lambda$ isotypic decomposition
  - partition $\lambda = (5, 3, 3, 1)$ for $n = 12$
- $V_\lambda = \bigoplus_{\tau_\lambda} W_{\tau_\lambda}$
  - shape of λ:

standard tableau τ_λ:

1	4	5	6	9
2	7	10		
3	8	12		
11				

  - $\mathfrak{R}_{\tau_\lambda}$:=row group of τ_λ (fixes the rows of τ_λ)
  - $W_{\tau_\lambda} := (V_\lambda)^{\mathfrak{R}_{\tau_\lambda}}$ = subspace of V_λ fixed by $\mathfrak{R}_{\tau_\lambda}$
  - n_λ:=number of standard tableaux of shape λ
  - m_λ:=dimension of W_{τ_λ}

Representation theory needed for exploiting symmetry

- $(\mathbb{R}[x]/\mathcal{I})_d =: V = \bigoplus_{\lambda \vdash n} V_\lambda$ isotypic decomposition
  - partition $\lambda = (5, 3, 3, 1)$ for $n = 12$
- $V_\lambda = \bigoplus_{\tau_\lambda} W_{\tau_\lambda}$
  - shape of λ:  standard tableau τ_λ:

1	4	5	6	9
2	7	10		
3	8	12		
11				

  - $\mathfrak{R}_{\tau_\lambda}$:=row group of τ_λ (fixes the rows of τ_λ)
  - $W_{\tau_\lambda} := (V_\lambda)^{\mathfrak{R}_{\tau_\lambda}}$ = subspace of V_λ fixed by $\mathfrak{R}_{\tau_\lambda}$
  - n_λ:=number of standard tableaux of shape λ
  - m_λ:=dimension of W_{τ_λ}

$$\boxed{V = \bigoplus_{\lambda \vdash n} \bigoplus_{\tau_\lambda} W_{\tau_\lambda}}$$

Note: $\dim(V) = \sum_{\lambda \vdash n} m_\lambda n_\lambda$

Gatermann-Parrilo symmetry-reduction technique

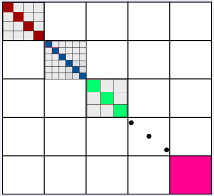
Recall: p d-sos mod $\mathcal{I} \Leftrightarrow \exists\ Q \succeq 0$ s.t. $p \equiv v^\top Q v$ mod $\mathcal{I}$
where v =vector of basis elements of $(\mathbb{R}[x]/\mathcal{I})_d$

Theorem (Gatermann-Parrilo, 2004)

For each λ, fix τ_λ and find a symmetry-adapted basis $\{b_1^{\tau_\lambda}, \ldots, b_{m_\lambda}^{\tau_\lambda}\}$ for W_{τ_λ}.

If p is symmetric and d-sos mod $\mathcal{I}$, then

$$p \equiv \sum_{\lambda \vdash n} \mathrm{sym}(b^\top Q_\lambda b) \quad \text{mod } \mathcal{I},$$



where $b = (b_1^{\tau_\lambda}, \ldots, b_{m_\lambda}^{\tau_\lambda})^\top$ and $Q_\lambda \succeq 0$ has size $m_\lambda \times m_\lambda$.

Gain: size of SDP is $\sum_{\lambda \vdash n} m_\lambda$ instead of $\sum_{\lambda \vdash n} m_\lambda n_\lambda$

Gatermann-Parrilo symmetry-reduction technique

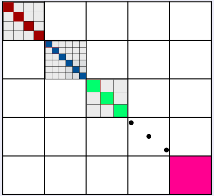
Recall: p d-sos mod $\mathcal{I} \Leftrightarrow \exists\ Q \succeq 0$ s.t. $p \equiv v^\top Q v$ mod $\mathcal{I}$
where v =vector of basis elements of $(\mathbb{R}[x]/\mathcal{I})_d$

Theorem (Gatermann-Parrilo, 2004)

For each λ, fix τ_λ and find a symmetry-adapted basis $\{b_1^{\tau_\lambda}, \ldots, b_{m_\lambda}^{\tau_\lambda}\}$ for W_{τ_λ}.

If p is symmetric and d-sos, then

$$p = \sum_{\lambda \vdash n} \mathrm{sym}(b^\top Q_\lambda b),$$



where $b = (b_1^{\tau_\lambda}, \ldots, b_{m_\lambda}^{\tau_\lambda})^\top$ and $Q_\lambda \succeq 0$ has size $m_\lambda \times m_\lambda$.

Gain: size of SDP is $\sum_{\lambda \vdash n} m_\lambda$ instead of $\sum_{\lambda \vdash n} m_\lambda n_\lambda$

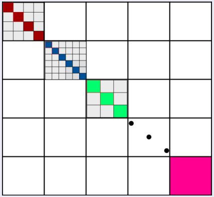
$\rightarrow$ how much smaller is the size of this SDP?

Gatermann-Parrilo symmetry-reduction technique

Recall: p d-sos mod $\mathcal{I} \Leftrightarrow \exists\ Q \succeq 0$ s.t. $p \equiv v^\top Q v$ mod $\mathcal{I}$
where v =vector of basis elements of $(\mathbb{R}[x]/\mathcal{I})_d$

Theorem (Gatermann-Parrilo, 2004)

For each λ, fix τ_λ and find a symmetry-adapted basis $\{b_1^{\tau_\lambda}, \dots, b_{m_\lambda}^{\tau_\lambda}\}$ *for* W_{τ_λ}.→ *complexity of the algorithm depends on n*



If p is symmetric and d-sos, then

$$p = \sum_{\lambda \vdash n} \mathrm{sym}(b^\top Q_\lambda b),$$

where $b = (b_1^{\tau_\lambda}, \dots, b_{m_\lambda}^{\tau_\lambda})^\top$ and $Q_\lambda \succeq 0$ has size $m_\lambda \times m_\lambda$.

Gain: size of SDP is $\sum_{\lambda \vdash n} m_\lambda$ instead of $\sum_{\lambda \vdash n} m_\lambda n_\lambda$

→ how much smaller is the size of this SDP?

Succinct SOS

Theorem (RSST, 2016)

If p is symmetric and d-sos, then it has a symmetry-reduced sos certificate that can be obtained by solving a SDP of size independent of n by keeping only a few partitions in Gatermann-Parrilo.

Succinct SOS

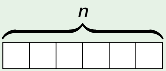
Theorem (RSST, 2016)

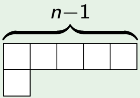
If p is symmetric and d-sos, then it has a symmetry-reduced sos certificate that can be obtained by solving a SDP of size independent of n by keeping only a few partitions in Gatermann-Parrilo.

Example

In the sos proof of Mantel's theorem

$$\begin{pmatrix} 1 & q_1 \end{pmatrix} \begin{pmatrix} \frac{(n-1)^2}{2} & -\frac{2(n-1)}{n} \\ -\frac{2(n-1)}{n} & \frac{8}{n^2} \end{pmatrix} \begin{pmatrix} 1 \\ q_1 \end{pmatrix} + \text{sym}\left(\begin{pmatrix} q_2 \end{pmatrix} \begin{pmatrix} \frac{8}{n^2} \end{pmatrix} \begin{pmatrix} q_2 \end{pmatrix} \right)$$

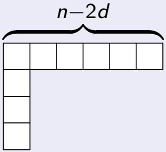
$\rightarrow$ kept partitions $(n) =$ 

and $(n-1, 1) =$ 

Bypassing symmetry-adapted basis

Theorem (RSST, 2016)

In Gatermann-Parrilo, instead of a symmetry-adapted basis, one can use

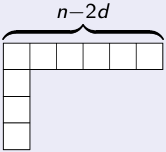
- *a spanning set for* W_{τ_λ} *for* $\lambda \geq_{\text{lex}}$ 

.
- *of size independent of n*
- *that is easy to generate*

Bypassing symmetry-adapted basis

Theorem (RSST, 2016)

In Gatermann-Parrilo, instead of a symmetry-adapted basis, one can use

- *a spanning set for* W_{τ_λ} *for* $\lambda \geq_{\text{lex}}$ 

.
- *of size independent of n*
- *that is easy to generate*

Examples of spanning sets containing W_{τ_λ}

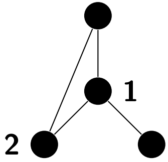
- $\text{sym}_{\tau_\lambda}(x^m) := \frac{1}{|\mathfrak{R}_{\tau_\lambda}|} \sum_{\mathfrak{s} \in \mathfrak{R}_{\tau_\lambda}} \mathfrak{s} \cdot x^m$
- an appropriate Möbius transformation

Razborov's flag algebras for Turán-type problems

Use flags (=partially labelled graphs) to certify a symmetric inequality that gives a good upper bound for Turán-type problems

Key features:

- sums of squares of graph densities
- n disappears
- asymptotic results for dense graphs



Theorem (Razborov, 2010)

If $\mathcal{A} = \{K_4^3\}$, *then* $\max_{G:|V(G)|\to\infty} d(G) \leq 0.561666$.
If $\mathcal{A} = \{K_4^3, H_1\}$, *then* $\max_{G:|V(G)|\to\infty} d(G) = 5/9$.

Complexity Theory at Oberwolfach in 2015

"Is there a link between sums of squares theory and flag algebras?"

Complexity Theory at Oberwolfach in 2015

"Is there a link between sums of squares theory and flag algebras?"

"No."

Connection of spanning sets to flag algebras

$$\tau_\lambda = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 & 1 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \to \text{hook}(\tau_\lambda) = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 \\ \cline{1-1} 1 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

g^{Θ} 

$$:= \text{sym}_{\text{hook}(\tau_\lambda)}(x_{12}x_{13}x_{14})$$

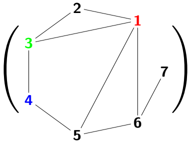
$$= \frac{1}{4}\left(x_{12}x_{13}x_{14} + x_{15}x_{13}x_{14} + x_{16}x_{13}x_{14} + x_{17}x_{13}x_{14}\right)$$

where $\Theta(1) = 1$, $\Theta(2) = 4$, $\Theta(3) = 3$, and g^{Θ} 

is the density of 

as a subgraph in some graph on 7 vertices under Θ.

Example: g^{Θ} 



Connection of spanning sets to flag algebras

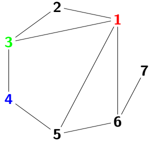
$$\tau_\lambda = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 & 1 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \rightarrow \text{hook}(\tau_\lambda) = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline {\color{green}3} \\ \cline{1-1} {\color{red}1} \\ \cline{1-1} {\color{blue}4} \\ \cline{1-1}\end{array}$$

g^{Θ} 

$$:= \text{sym}_{\text{hook}(\tau_\lambda)}(x_{{\color{red}1}2}x_{{\color{red}1}{\color{green}3}}x_{{\color{red}1}{\color{blue}4}})$$

$$= \frac{1}{4}\left(x_{{\color{red}1}2}x_{{\color{red}1}{\color{green}3}}x_{{\color{red}1}{\color{blue}4}} + x_{{\color{red}1}5}x_{{\color{red}1}{\color{green}3}}x_{{\color{red}1}{\color{blue}4}} + x_{{\color{red}1}6}x_{{\color{red}1}{\color{green}3}}x_{{\color{red}1}{\color{blue}4}} + x_{{\color{red}1}7}x_{{\color{red}1}{\color{green}3}}x_{{\color{red}1}{\color{blue}4}}\right)$$

where $\Theta(1) = {\color{red}1}$, $\Theta(2) = {\color{blue}4}$, $\Theta(3) = {\color{green}3}$, and g^{Θ}  is the density of  as a subgraph in some graph on 7 vertices under Θ.

Example: g^{Θ}  $\Big($  $\Big) = 0$

Connection of spanning sets to flag algebras

$$\tau_\lambda = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 & 1 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \rightarrow \text{hook}(\tau_\lambda) = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 \\ \cline{1-1} 1 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

g^{Θ} 

$$:= \text{sym}_{\text{hook}(\tau_\lambda)}(x_{12}x_{13}x_{14})$$

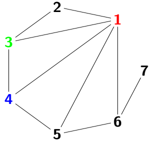
$$= \frac{1}{4}\left(x_{12}x_{13}x_{14} + x_{15}x_{13}x_{14} + x_{16}x_{13}x_{14} + x_{17}x_{13}x_{14}\right)$$

where $\Theta(1) = 1$, $\Theta(2) = 4$, $\Theta(3) = 3$, and g^{Θ} 

is the density of 

as a subgraph in some graph on 7 vertices under Θ.

Example: g^{Θ} 

$\Bigg($ 

$\Bigg)$

Connection of spanning sets to flag algebras

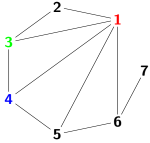
$$\tau_\lambda = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 & 1 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \rightarrow \text{hook}(\tau_\lambda) = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline \color{green}{3} \\ \cline{1-1} \color{red}{1} \\ \cline{1-1} \color{blue}{4} \\ \cline{1-1}\end{array}$$

g^{Θ} 

$$:= \text{sym}_{\text{hook}(\tau_\lambda)}(x_{12}x_{13}x_{14})$$

$$= \frac{1}{4}\left(x_{12}x_{13}x_{14} + x_{15}x_{13}x_{14} + x_{16}x_{13}x_{14} + x_{17}x_{13}x_{14}\right)$$

where $\Theta(1) = 1$, $\Theta(2) = 4$, $\Theta(3) = 3$, and g^{Θ}  is the density of  as a subgraph in some graph on 7 vertices under Θ.

Example: g^{Θ}  $\left(\right.$ 

$\left. \right) = \frac{3}{4}$

Connection of spanning sets to flag algebras

$$\tau_\lambda = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 & 1 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \rightarrow \text{hook}(\tau_\lambda) = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 \\ \cline{1-1} 1 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

g^{Θ} 

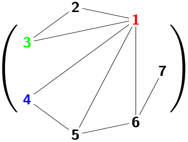
$$:= \text{sym}_{\text{hook}(\tau_\lambda)}(x_{12}x_{13}x_{14})$$
$$= \frac{1}{4}(x_{12}x_{13}x_{14} + x_{15}x_{13}x_{14} + x_{16}x_{13}x_{14} + x_{17}x_{13}x_{14})$$

where $\Theta(1) = 1$, $\Theta(2) = 4$, $\Theta(3) = 3$, and g^{Θ} 

is the density of 

as a subgraph in some graph on 7 vertices under Θ.

Example: g^{Θ} 



Connection of spanning sets to flag algebras

$$\tau_\lambda = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 & 1 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \rightarrow \text{hook}(\tau_\lambda) = \begin{array}{|c|c|c|c|}\hline 2 & 5 & 6 & 7 \\ \hline 3 \\ \cline{1-1} 1 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

g^{Θ} 

$$:= \text{sym}_{\text{hook}(\tau_\lambda)}(x_{12}x_{13}x_{14})$$

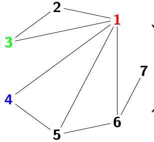
$$= \frac{1}{4}\left(x_{12}x_{13}x_{14} + x_{15}x_{13}x_{14} + x_{16}x_{13}x_{14} + x_{17}x_{13}x_{14}\right)$$

where $\Theta(1) = 1$, $\Theta(2) = 4$, $\Theta(3) = 3$, and g^{Θ} 

is the density of 

as a subgraph in some graph on 7 vertices under Θ.

Example: g^{Θ} 

$\Bigg($ 

$\Bigg) = \frac{3}{4}$

Connection of spanning sets to flag algebras

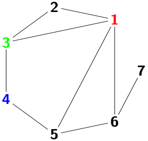
Möbius transformation $\rightarrow$ d^{Θ} 

: density of 

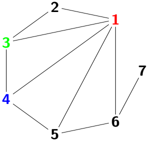
as an *induced* subgraph in some graph on 7 vertices under Θ such that $\Theta(1) = 1$, $\Theta(2) = 4$, $\Theta(3) = 3$ $\rightarrow$ flag density.

Example:

d^{Θ} 

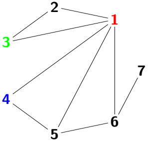
$\Bigg($ 

$\Bigg) = 0,$ d^{Θ} 

$\Bigg($ 

$\Bigg) = 0,$

and d^{Θ} 

$\Bigg($ 

$\Bigg) = \frac{1}{4}$

Connection of spanning sets to flag algebras

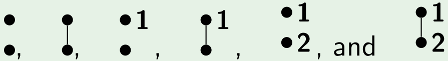
Theorem (RSST, 2016)

Flags provide spanning sets for W_{τ_λ} of size independent of n.

If p is symmetric and d-sos, then its nonnegativity can be established through flags on kd vertices (even in restricted cases).

Example

For the sos proof of Mantel's theorem, need at most flags:



Connection of spanning sets to flag algebras

Theorem (R., Singh, Thomas, 2015)

Every flag sos polynomial of degree kd can be written as a succinct d-sos.

Theorem (RSST, 2016)

Flag methods are equivalent to standard symmetry-reduction methods for finding sos certificates over discrete hypercubes.

Consequences of this connection

Corollary (RSST, 2016)

It is possible to use flags for a fixed n, not just asymptotic situations

Example

The following flag sos yields the Ramsey number $R(3,3) \leq 6$

$$-1 \equiv \frac{1}{8\binom{6}{2}^2}\left(\mathsf{d}^{\Theta}_{\bullet\!-\!\bullet} + \mathsf{d}^{\Theta}_{\bullet\;\bullet}\right)^2 + \mathbb{E}_{\Theta_i}\left[\frac{1}{2}\left(\mathsf{d}^{\Theta_i}_{\bullet\!-\!\bullet_1} - \mathsf{d}^{\Theta_i}_{\bullet\;\bullet_1}\right)^2\right] \mod \mathcal{I}$$

where

$$\mathsf{d}^{\Theta}_{\bullet\!-\!\bullet} = 2\sum_{1\leq i<j\leq 6} \mathsf{x}_{ij}, \qquad \mathsf{d}^{\Theta}_{\bullet\;\bullet} = 2\sum_{1\leq i<j\leq 6}(1-\mathsf{x}_{ij}),$$

$$\mathsf{d}^{\Theta_i}_{\bullet\!-\!\bullet_1} = \sum_{j\in[6]\setminus\{i\}} \mathsf{x}_{ij}, \qquad \mathsf{d}^{\Theta_i}_{\bullet\;\bullet_1} = \sum_{j\in[6]\setminus\{i\}}^{6}(1-\mathsf{x}_{ij})$$

Consequences of this connection

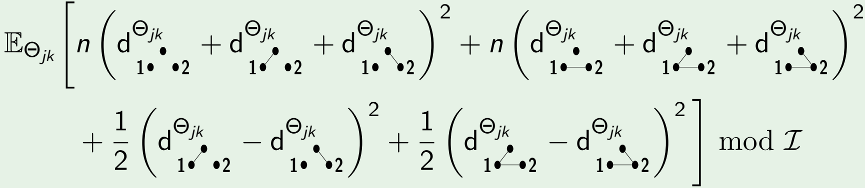
Corollary (RSST, 2016)

It is possible to use flags for extremal graph theoretic problems in the sparse setting.

Example

The following flag sos yields that the max edge density in C_4-free graphs is at most $\frac{n^{3/2}}{n^2-n} + O\left(\frac{1}{n}\right)$ (Sós et al)

$$n + \frac{2}{n-1}\mathrm{s} - \frac{2}{\binom{n}{2}}\mathrm{s}^2 \equiv$$



Consequences of this connection

Example (Grigoriev's family of polynomials, 2001)

The polynomials

$$\mathrm{f}_n = \frac{1}{\binom{n}{2}^2}\left(\sum_{e\in E(K_n)} \mathrm{x}_e - \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor\right)\left(\sum_{e\in E(K_n)} \mathrm{x}_e - \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor - 1\right)$$

are nonnegative on $\mathcal{V}_{n,2}$.

The degree required to write f_n as a SOS is at least $\left\lceil \frac{\binom{n}{2}}{2} \right\rceil$

Certifying nonnegativity $f_n + O(\frac{1}{n^2})$ also requires an SOS of degree $\left\lceil \frac{\binom{n}{2}}{2} \right\rceil$

(Lee, Prakesh, de Wolf, Yuen, 2016)

Consequences of this connection

Hatami-Norin (2011) showed that the nonnegativity of graph density inequalities in general is undecidable

Corollary (RSST, 2016)

There exists a family of symmetric nonnegative polynomials of fixed degree that cannot be certified with any fixed set of flags, namely

$$\frac{1}{\binom{n}{2}^2}\left(\sum_{e\in E(K_n)} x_e - \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor\right)\left(\sum_{e\in E(K_n)} x_e - \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor - 1\right) + O(\frac{1}{n^2})$$

Note: Razborov allows error of size $O(\frac{1}{n})$ in his setting

Open problems

- Find a concrete family of nonnegative polynomials on $\binom{n}{k}$ variables that one cannot approximate up to an error of order $O(\frac{1}{n})$ with finitely many flags or with sums of squares of fixed degree.
- Provide certificates for open problems over $\mathcal{V}_{n,k}$ using symmetric sums of squares.

Open problems

- Find a concrete family of nonnegative polynomials on $\binom{n}{k}$ variables that one cannot approximate up to an error of order $O(\frac{1}{n})$ with finitely many flags or with sums of squares of fixed degree.
- Provide certificates for open problems over $\mathcal{V}_{n,k}$ using symmetric sums of squares.

Thank you!