Mogami triangulations

Bruno Benedetti

University of Miami

MSRI, october 10, 2017

Shellable complexes

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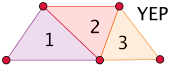

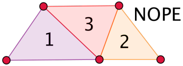

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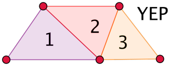

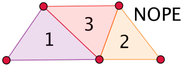

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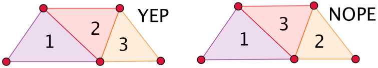
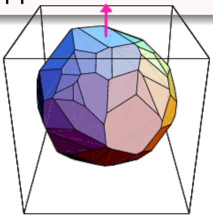


Fact (Bruggesser–Mani 1971, also Schläfli 1850)

The boundaries of simplicial polytopes are shellable: "Lift off" from a facet, and moving along a generic line, record the facets in the order in which they appear at the horizon ("rocket shelling").



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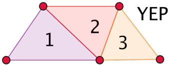

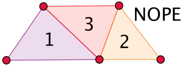

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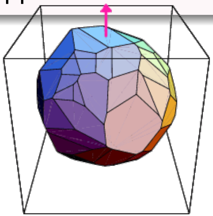
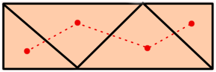


Trees of d-simplices



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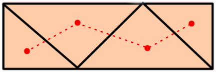

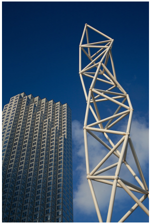

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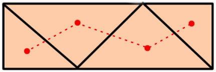

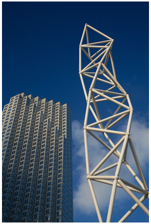

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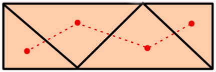

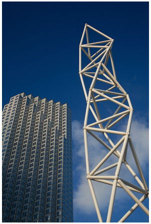

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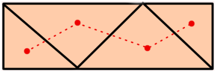

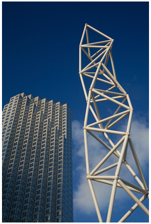

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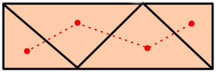

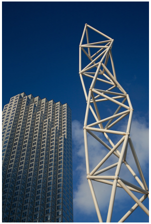

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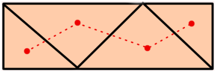

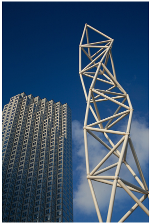

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$$C_d(N) = \frac{1}{(d-1)N+1}\binom{dN}{N}$$

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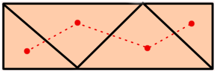

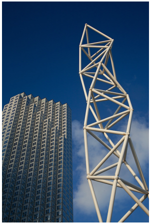

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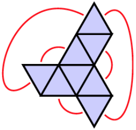

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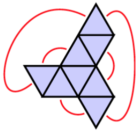

Figure: Gluing boundary edges according to red matching ⇝ octahedron

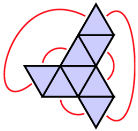

Figure: Gluing boundary edges according to red matching $\rightsquigarrow$ octahedron

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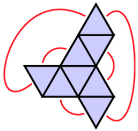

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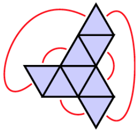

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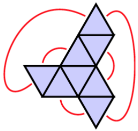

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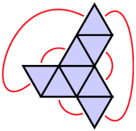

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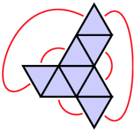

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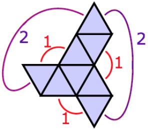
What if we want an exponential bound (in N, for fixed d)

From now on, $d \geq 2$.



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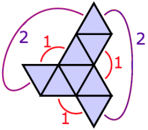

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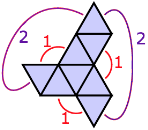

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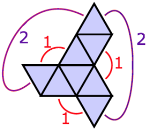

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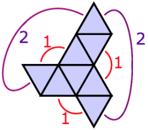

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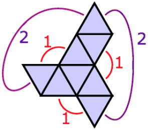

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Theorem [Durhuus–Jonsson 1995, Mogami 1995, B.–Ziegler 2011]

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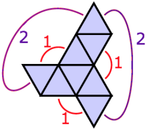

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In dimension $d \leq 3$, also Mogami d-manifolds are exp. many.

Connection to topology

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Every Mogami manifold is simply connected.

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So LC/Mogami triangulations = combinatorial way to capture simple connectedness (for manifolds).

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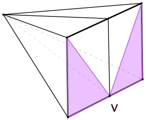
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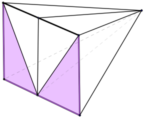

Gluing Lemma

Let A, B be two d-pseudomanifolds. Let $C \subset \partial A$ be a pure $(d-1)$-dimensional complex combinatorially equivalent to a subcomplex $C' \subset \partial B$. Let $A \cup B$ be the complex obtained from the disjoint union $A \sqcup B$ by identifying $C \equiv C'$.

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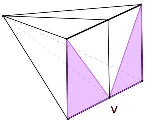

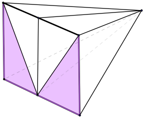

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- If A, B are LC and C is strongly connected, $A \cup B$ is LC.

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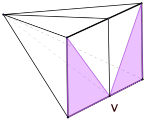

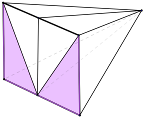

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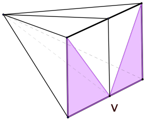

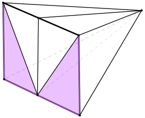

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If any is true, then 3-balls are exponentially many!

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... unfortunately, this characterization does not extend to Mogami.
So Mogami's conjecture stayed open.

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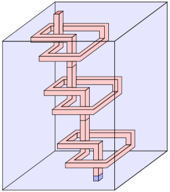

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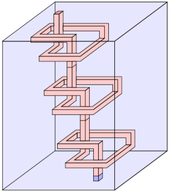

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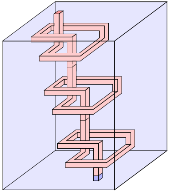

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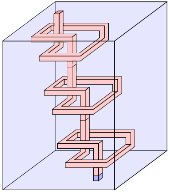

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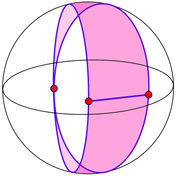
$\rightsquigarrow$ New idea: The Mogami construction of a 3-ball without interior vertices, could be spartan... 'Chic' gluings may cost interior vertices!

Example of gluing that we cannot afford

Say that in constructing a 3-ball, after some Mogami steps (which could take us out of the world of simplicial complexes), we see two boundary triangles that share one edge plus the opposite vertex. What happens if we decide to glue them?



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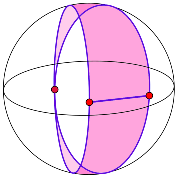

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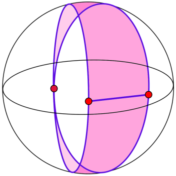

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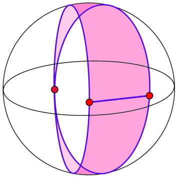

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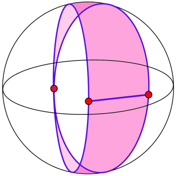

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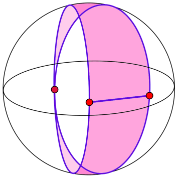

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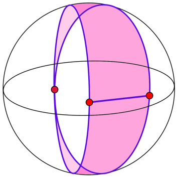

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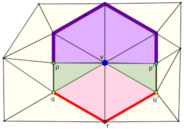
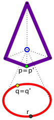
Imagine in the boundary of a finely triangulated 3-ball, we see this:





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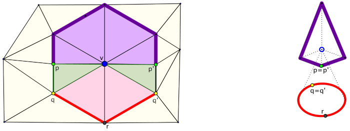

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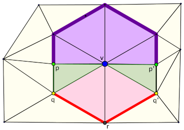

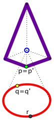

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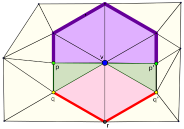

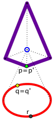

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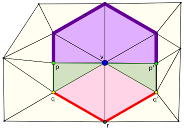

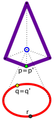

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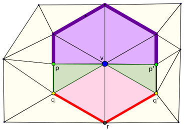

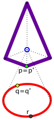

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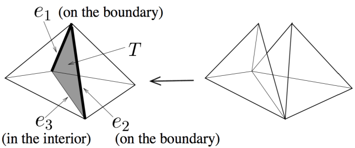

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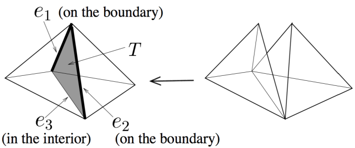

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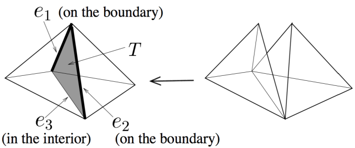

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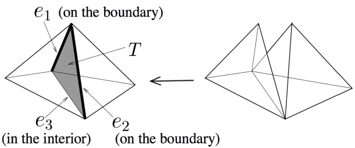

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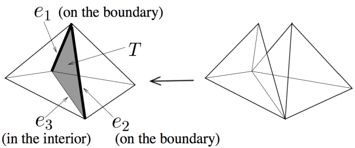

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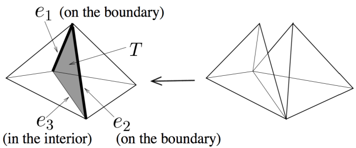

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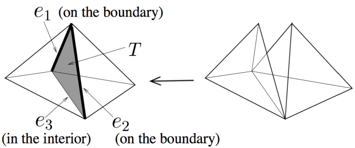

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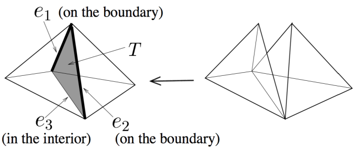

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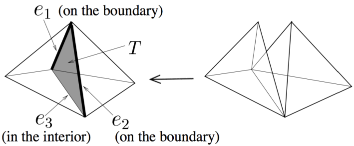

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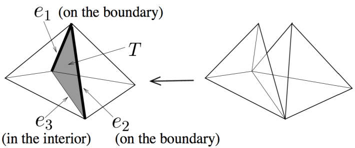
Every 3-ball with knotted spanning edges and without interior vertices, is not Mogami.

Main Result

Theorem (B. 2017)

Every Mogami 3-ball without interior vertices is obtainable from a tree of tetrahedra using only FOLD moves.



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Note: This behavior is completely different for the LC property.

Theorem [B.–Ziegler, 2011]

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So the cone over an annulus, say, is Mogami but not LC. (Annulus is not simply connected, so not Mogami, so not LC.)

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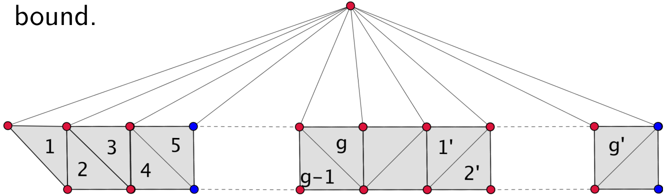

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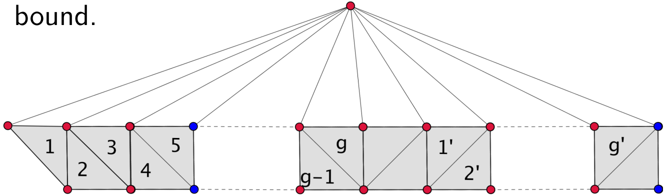

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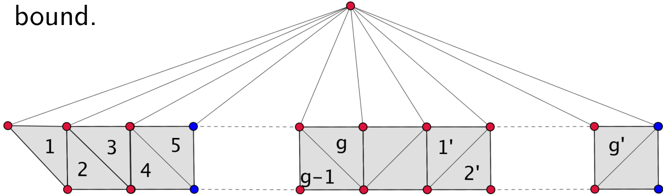

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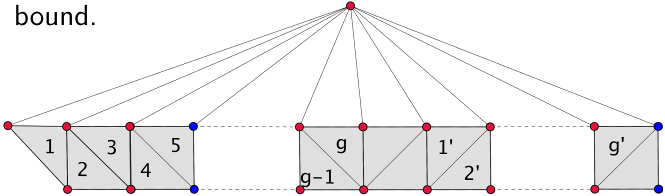

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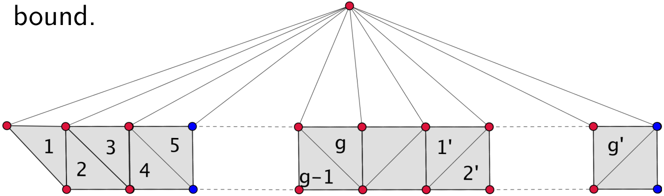

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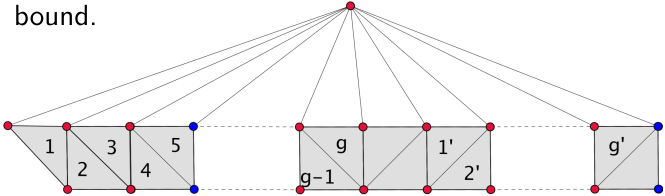

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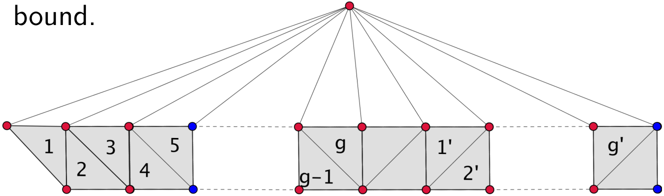

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