The Geometry of Scheduling

Caroline J. Klivans
Brown University

Joint work with Felix Breuer

A Scheduling Problem

n jobs to be performed:

$$x_1$$

$$x_2$$

$$x_3$$

$$\vdots$$

With constraints of the form:

$$x_i \leq x_j$$

- $x_1 \leq x_2 \leq \cdots \leq x_n$ specified linear order
- $(x_1 \leq x_5) \vee (x_3 \leq x_5)$ job 5 can't be done until job 1 or job 3

Valid Schedules

Schedule the n jobs using at most t time slots.

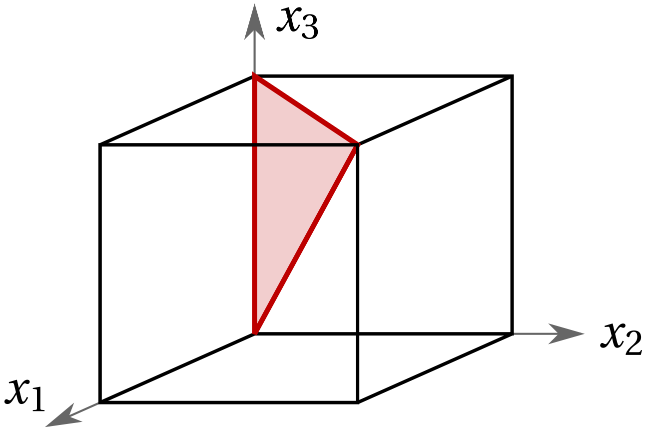
Let S be the boolean function of constraints.
A ***t*-schedule** solving S is an integer assignment

$$\omega : [n] \to [t]$$

which makes $S(\omega)$ true.

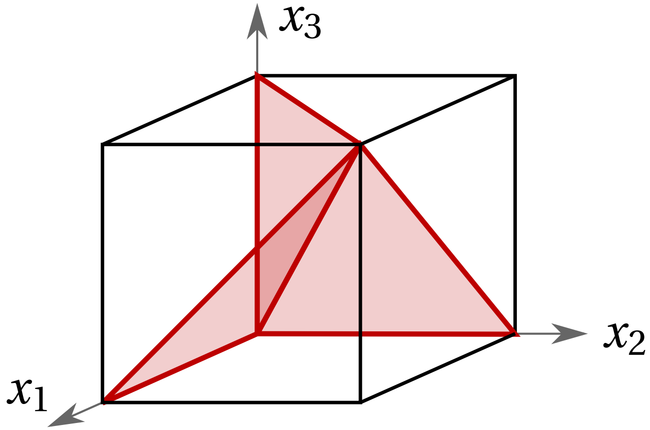
- $(x_1 \le x_2 \le x_3)$ $(1,1,1),(1,1,2),(1,2,2),(3,5,5),\ldots$
- $(x_1 \le x_5) \vee (x_3 \le x_5)$ $(1,2,8,8,4),(8,2,1,8,4),\ldots$

Schedules Geometrically



If jobs 1 and 2 run at the same time, then job 3 has to run first.

$$x_1 = x_2 \Rightarrow x_3 < x_1$$



If two jobs run rimultaneously, the third job has to run first.

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Counting Valid Schedules

Define $p_S(t)$ = the number of t-schedules solving S.

Theorem BK

$$p_S(t) \text{ is a polynomial function in } t.$$

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Scheduling polynomials include:

- chromatic polynomial of graph
- zeta polynomial of a lattice
- Billera-Jia-Reiner polynomial of a matroid
- arboricity polynomial of a matroid
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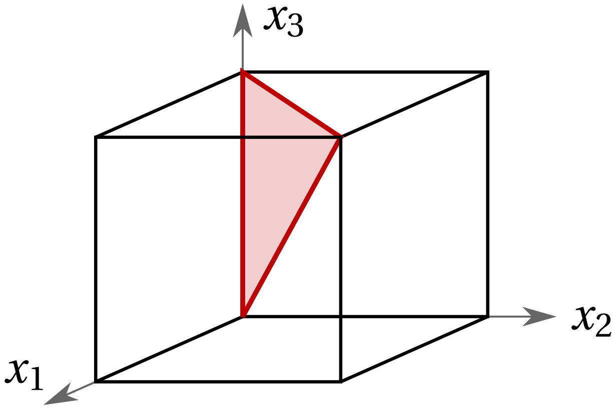
Hopf
Monoids

Ehrhart Polynomials



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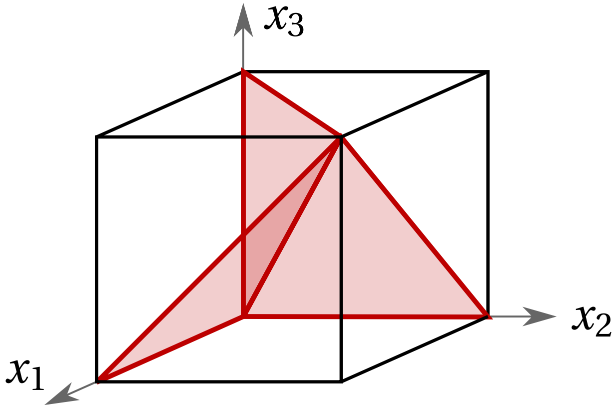

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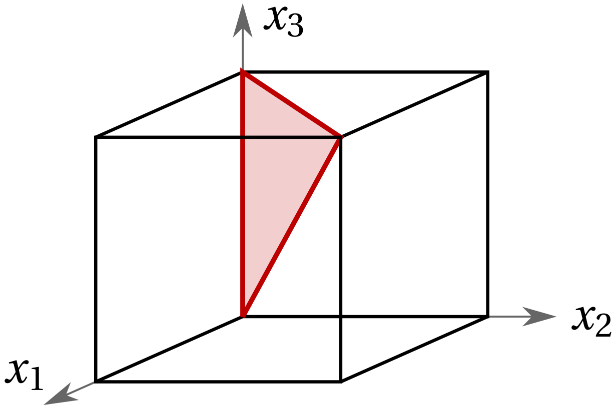
Schedules are *almost* the integer points inside a polytope.

Ehrhart Polynomials



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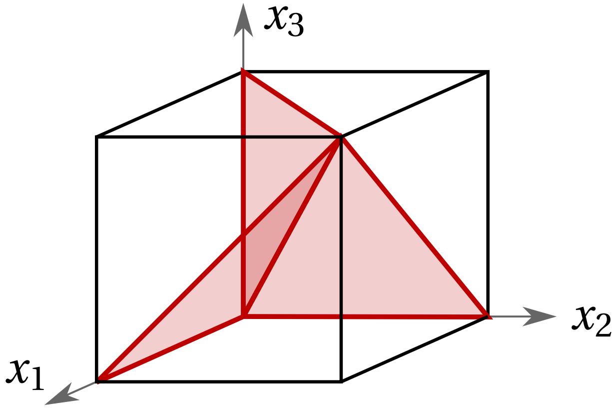

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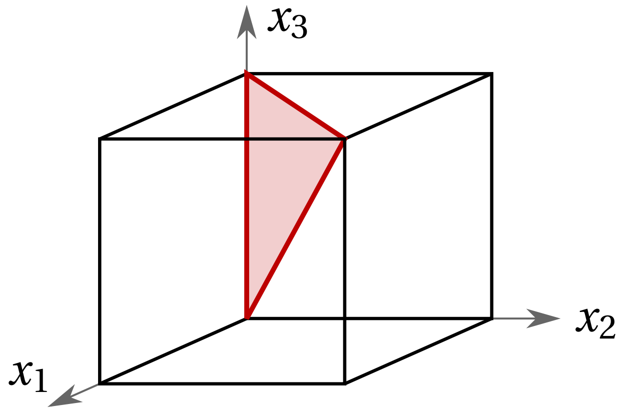

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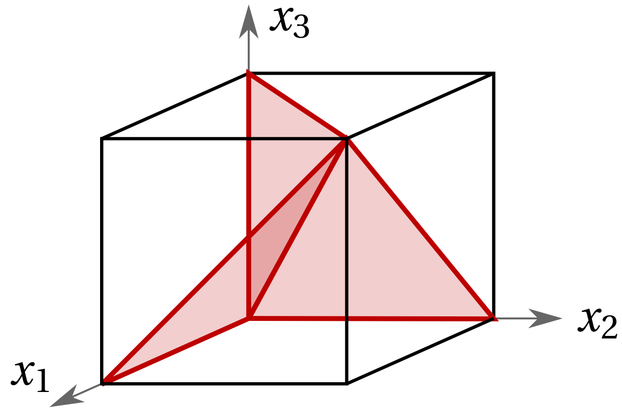

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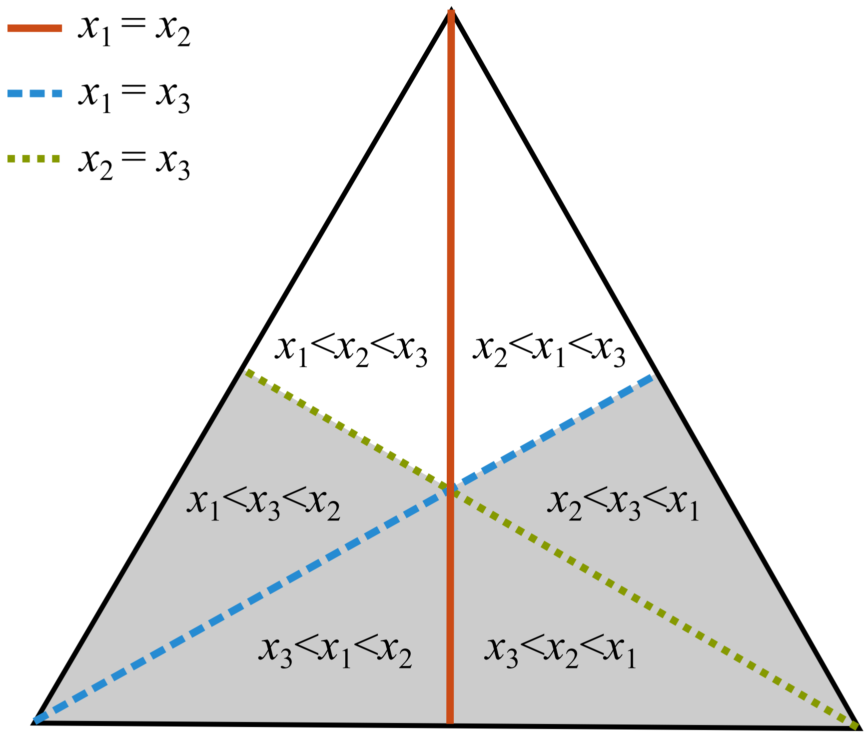
Integer points of inside-out polytopes. Beck-Zaslavsky

Braid Arrangement

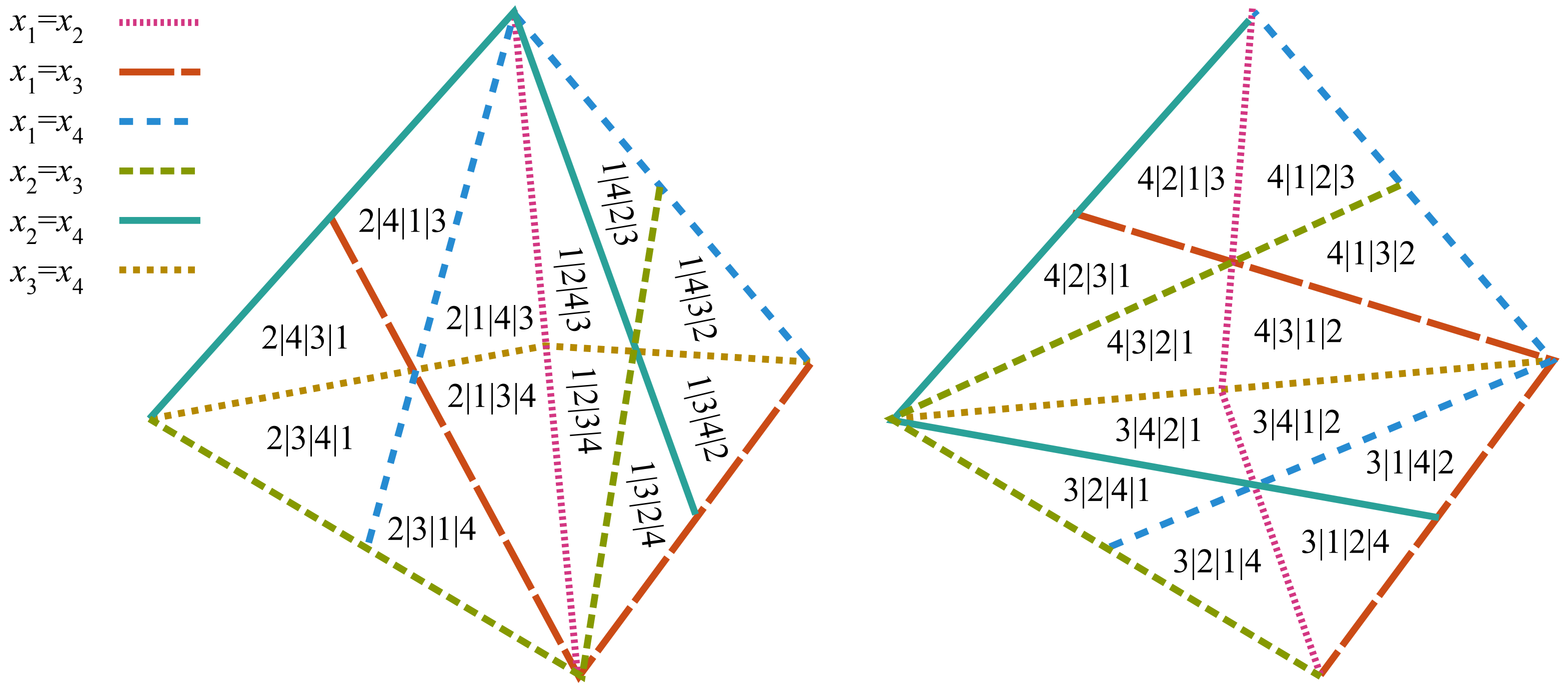
Braid arrangement $\mathcal{H} \in \mathbb{R}^n,\ \mathcal{H} := \{x_i = x_j,\ ij \in [n]\}$



If $x_1 < x_2$
then
    $x_3 < x_2$
else
    $x_3 < x_1$.

Coxeter Complex

Braid intersected with the sphere



Gives relative ordering of coordinates

$$x_2 = x_3 < x_1 = x_4 = x_6 < x_5$$

23|146|5 (ordered set partition)

NCQSym

Quasisymmetric functions in non-commuting variables.

Formal power series in non-commuting variables $\{x_1, x_2, \ldots\}$.

The coefficient of $x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_n}$ must equal the coefficient of $x_{\tau_1} x_{\tau_2} \cdots x_{\tau_n}$ if the relative order of γ and $\tau \in \mathbb{N}^n$ are the same.

$$3x_1x_2x_1x_2 + 3x_1x_3x_1x_3 + 3x_2x_3x_2x_3 + 3x_3x_4x_3x_4 + \cdots$$

$$(1,2,1,2),(1,3,1,3),(2,3,2,3),(3,4,3,4) \in \mathbb{N}^4$$

$$\updownarrow$$

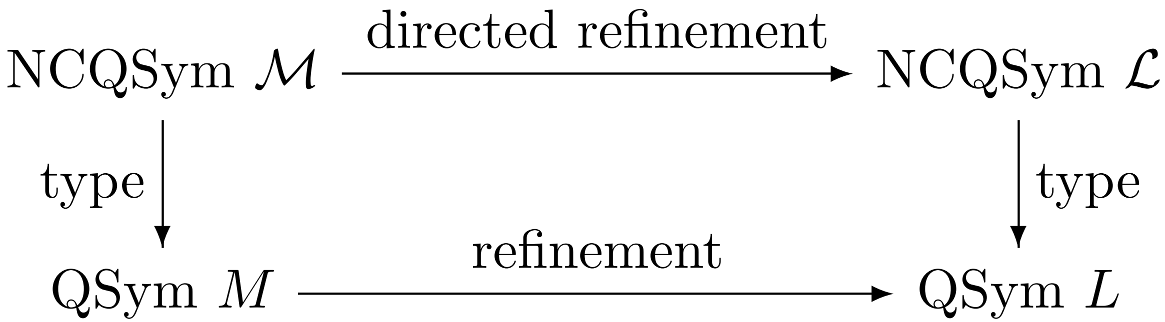
$13|24$ ordered set partition

NCQSym Monomials

$$\mathcal{M}_{13|2} = \sum_{0<i_3=i_1<i_2} x_{i_1} x_{i_2} x_{i_3}$$

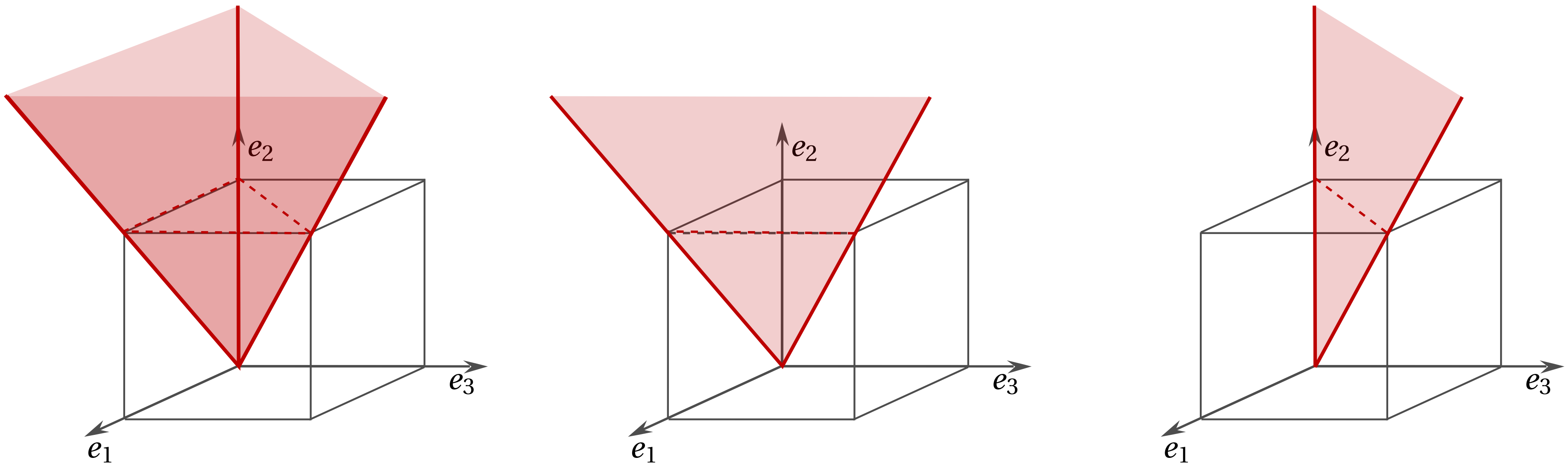
NCQSym $\to$ **Quasisymmetric functions**

$$\mathrm{type}(\Phi_1|\Phi_2|\cdots|\Phi_n) = (|\Phi_1|, |\Phi_2|, \cdots, |\Phi_n|) \quad \mathrm{type}(13|24) = (2,2)$$



NQSym Scheduling

t-schedules where $t \to \infty$, "there is no deadline."



$$\mathcal{M}_{3|1|2} = \sum_{0<i_3<i_1<i_2} x_{i_1} x_{i_2} x_{i_3} \qquad \mathcal{M}_{3|12} = \sum_{0<i_3<i_1=i_2} x_{i_1} x_{i_2} x_{i_3} \qquad \mathcal{M}_{13|2} = \sum_{0<i_3=i_1<i_2} x_{i_1} x_{i_2} x_{i_3}$$

NCQSym $\to$ Polynomials

Set first t variables equal to 1, others equal to 0.

$$\mathcal{M}_{\Phi}(\mathbf{1}^t) = \binom{t}{\ell(t)}$$

$$\mathcal{S} = \mathcal{M}_{1|23} + \mathcal{M}_{3|21} + \mathcal{M}_{2|1|3}$$

$$\mathcal{S}(\mathbf{1}^t) = 2\binom{t}{2} + \binom{t}{3}$$

Scheduling polynomial

$$p_s(t) = \mathcal{S}(\mathbf{1}^t) = \sum_{\Phi:S(\Phi)} \mathcal{M}_{\Phi}(\mathbf{1}^t) = \sum_{\Phi:S(\Phi)} \mathrm{ehr}_{\mathrm{cone}(\Phi)\cap(0,1)^n}(t+1)$$

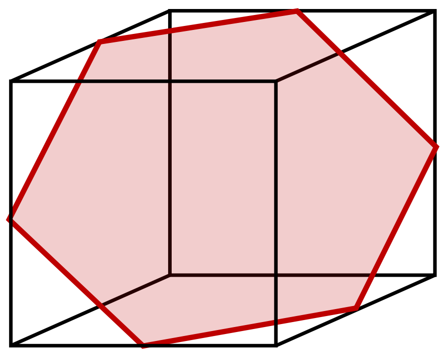
Graph Coloring

Graph $G = (V, E)$

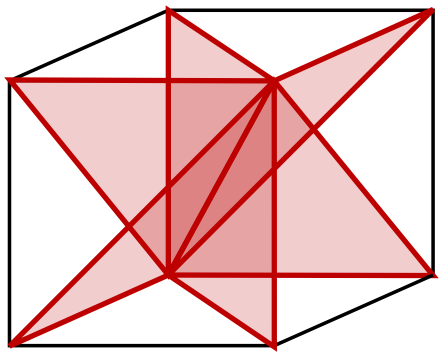
$$t\text{-coloring: } \phi(G) : V \to [t], \quad \phi(v_i) \neq \phi(v_j), \quad ij \in E$$

All integer points off of the planes $(x_i = x_j)$, $ij \in E$

(Relative interiors of graphical arrangement.)



graphical zonotope



graphical arrangement

Graph Coloring

Chromatic NCQSym(K_3)

$$= \mathcal{M}_{1|2|3} + \mathcal{M}_{2|1|3} + \mathcal{M}_{2|3|1} + \mathcal{M}_{1|3|2} + \mathcal{M}_{3|1|2} + \mathcal{M}_{3|2|1}$$

$$p_{K_3}(t) = t(t-1)(t-2)$$

Chromatic NCQSym Gebhard - Sagan
Chromatic QSym (from type map) Stanley
Chromatic polynomial (from specialization)

Note: Didn't use contraction/deletion.

Arboricity

The **Arboricity** of a graph / matroid, Nash-Williams, Tutte, Edmonds

- $a(G)$ = min number of forests to cover the edges of G
- $a(M)$ = min number of independent sets to cover ground set E of M

$$a(M) = \max_{X \subseteq E} \lceil \frac{|X|}{rk(X)} \rceil$$

Constructive version: matroid partitioning problem

Independent cover = ordered set partition of E s.t. no block contains a circuit

Arboricity as Scheduling

$$A_M = \bigwedge_{C \in \mathcal{C}} \neg(x_{i_1} = x_{i_2} = \cdots = x_{i_m})$$

"The collection of jobs that start at a fixed time can not be dependent."

Arboricity polynomial

$p_{A_M}(t) =$ number of independent covers with at most t parts.

- M free matroid: $p_{A_M}(t) = t^n$
- $M(C_n)$ graphical matroid of cycle graph: $p_{A_M}(t) = t^n - t$

Deletion/Contraction does not hold for arboricity polynomials.

Not a Tutte invariant.

Geometry of Scheduling

An ordered set partition $\Phi \longleftrightarrow$ face σ_Φ of Coxeter complex

Allowed and **Forbidden** configurations

Allowed: $\Lambda(S) = \{\sigma_\Phi \mid S(\Phi) \text{ is true}\}$

Forbidden: $\Gamma(S) = \{\sigma_\Phi \mid S(\Phi) \text{ is false}\}$

Example: Graph Coloring

Forbidden configuration $\Gamma(S)$ = Steingrimsson's coloring complex

Allowable configuration $\Lambda(S)$ = interiors of maximal cones of the normal fan of the graphical zonotope.

Steingrimsson Theorem

Hilbert polynomial of the chromatic ideal = chromatic polynomial

Scheduling - h-vectors

Theorem BK For any scheduling problem S:

The h-polynomial of Λ = scheduling polynomial

The h-polynomial of Γ = tail of scheduling polynomial

$$h(p_S(k-1)) = h(\Lambda(S))$$

$$h((k-2)^n - p_S(k-1)) = h(\Gamma(S))$$

Equivalently,

$$1 + t\sum_{k\geq 0} p_S(k)t^k = \frac{h_{(\Lambda(S))}(t)}{(1-t)^{d+1}}$$

$$1 + t\sum_{k\geq 0}((k-1)^n - p_S(k))t^k = \frac{h_{(\Gamma(S))}(t)}{(1-t)^{d+1}}$$

Bridge

Properties of geometric spaces $\longleftrightarrow$ Properties of algebraic invariants

Example: Partitionability and Positivity

- If $\Lambda(S)$ is partitionable then
Scheduling QSym is L-positive (fundamental basis)
Scheduling polynomial is h^*-positive

- If Scheduling NCQSym is $\mathcal{L}$-positive then
$\Lambda(S)$ is partitionable

- If S is a decision tree (nested if-then-else structure) then
$\Lambda(S)$ is partitionable

Matroid Polytopes

Allowable configuration Λ = interiors of maximal cones of the normal fan of a matroid polytope.

Forbidden configuration Γ = codimension one skeleton of the normal fan as subdivided by the Coxeter complex.

Schedules: For each vertex v, take conjunction of hyperplanes meeting at v.

Scheduling NCQSym: Φ = Generic M-weightings

Scheduling QSym: Billera-Jia-Reiner quasisymmetric function

Scheduling Polynomial: Counts the number of generic M-weightings

Graphic zonotope $\rightarrow$ Matroid polytope $\rightarrow$ Generalized Permutahedron

Scheduling - Flats

Rank as cost function. (Rank function of a Matroid)

"Once certain jobs are started, others in the closure can be added without additional cost. Minimize cost by requiring that in any scheduling of jobs, at each step we have a closed subset of jobs."

Φ is allowable if flag(Φ) is a flag of flats.

$$(x_7 < x_4 < x_3 < x_1 = x_2 = x_5 < x_6) \longleftrightarrow 7|4|3|125|6$$

$$\longleftrightarrow 7 \subset 47 \subset 347 \subset 123457 \subset 1234567$$

On the matroid polytope:

Weights ω: Every element is in some ω-min basis.

$$\omega = (4, 4, 3, 2, 4, 5, 1)$$

Scheduling - Bergman

Allowable $\Lambda(M)$ = Bergman complex of M (tropical linear spaces)

Scheduling Qsym: Bergman quasisymmetric function

Scheduling polynomial: Counts the number of every-element-minimizing M-weightings

$$p_M(t) = Z(L(M), t) = |\{\hat{0} = y_0 \leq y_1 \leq \cdots \leq y_t = \hat{1} | y_i \in L(M)\}|$$

Zeta-polynomial of the lattice of flats of M.

Counts multi-chains of length t in lattice of flats.