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Around the lower bound theorem for polytopes

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A. Classics

P = simplicial d-dimensional polytope on n vertices

$f_i(P)$ = # of i-dim faces

Lower bound theorem (LBT):

① (Barnette '71, '73)

$\forall i \quad f_i(P) \geq f_i(S(d,n)) = \text{func}(d,n,i)$

↑ stacked polytopes

② (Kalai '87)

$$\underset{\displaystyle f_1 - dn + \binom{d+1}{2}}{\overset{\displaystyle g_2}{\shortparallel}} = 0 \iff P = S(d,n)$$

Kalai's observation: Rigidity ⇒ LBT

Infinitesimal Rigidity

$G = (V,E)$ simple finite graph

$\varphi : V \longrightarrow \mathbb{R}^d$ (locations)

$\alpha : V \longrightarrow \mathbb{R}^d$ (velocities) preserving edge length up to first order, i.e.

$$(\star) \quad \forall uv \in E \quad \frac{d}{dt} \| (\varphi(v) + t\alpha(v)) - (\varphi(u) + t\alpha(u)) \|^2 \Big|_{t=0} = 0$$

Def: φ is infinitesimally rigid if $\forall$ such α, $(\bigstar)$ holds $\forall \underset{uv}{u \neq v} \in V$

Eq. the rigidity matrix $R(G,P)_{d|V| \times |E|} = \begin{array}{c} v - \\ \\ u - \end{array} \begin{pmatrix} \varphi(v)-\varphi(u) \\ \\ \varphi(u)-\varphi(v) \end{pmatrix}$

has rank $= \operatorname{rank} R(K_V, \varphi)$

* For generic φ, rank is maximized

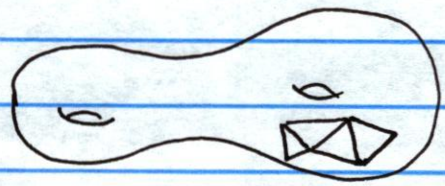
** and if $|V| = m > d$ then $\operatorname{rank} R(K_V) = R(K_V, \varphi) =$
$= dm - \binom{d+1}{2}$.

Def: G is generically d-rigid if a generic φ is infinitesimally rigid.

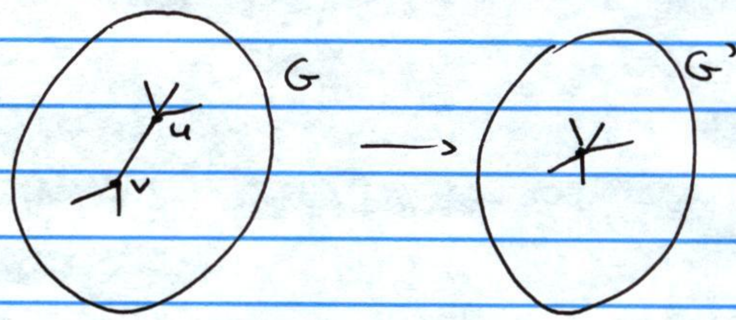
Alexandrov '50: $\forall d \geq 3$, $P_{\leq 2}$ defines infinitesimally rigid φ.

B. Generalizations

Fogelsanger '88: Δ minimal $(d-1)$-dim cycle complex $(d \geq 3)$ over an Abelian group
$\Rightarrow$ $\Delta_{\leq 1}$ is generically d-rigid.

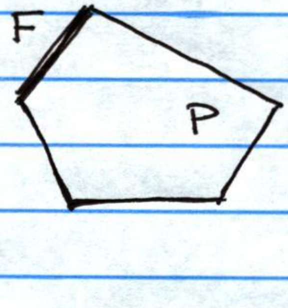


Whitley's vertex splitting:



if u, v have $\geq d-1$ common neighbors
then G' generically d-rigid
$\Rightarrow$ G generically d-rigid.

Almost simplicial polytopes:



F facet of P with $d+s$ vertices
$\partial P \setminus \{F\}$ simplicial ball B
($\Rightarrow$ ∂B is induced subcomplex)

Lower bound theorem for almost simplicial polytopes:

LBT (N. - Pineda Villavicencio - Ugon - Yost)

① $\forall i \quad f_i(P) \geq f_i(S(n,d,s)) = \text{func}(n,d,i,s)$

$f_i(S(n,d)) - (0\ldots 0,s,s)$ stack or H-stack

② equality for some $i > 0$ iff · if $d \geq 5$ $P = S(n,d,s)$

· if $d = 4$ $\exists$ characterization ...

For any polytope, $g_2(P) = f_1 - df_0 + \binom{d+1}{2} + \sum_{\substack{F \subseteq P \\ \text{2-face}}} (f_0(F) - 3)$

Whiteley '84: $g_2(P) \geq 0$

Kalai's monotonicity '94: $g(P) \geq g(P/F) \cdot g(F)$

$\rightsquigarrow g_2(P) \geq g_2(F)$.

Related open problems:

· Is $g_2(Q) \geq 0$ for Q = polyhedral 3-spheres?

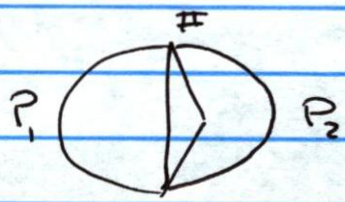
· Characterize $(f(P), f(F))$ for almost simplicial pairs (P, F).

· What are $\{P : g_2(P) = 0\}$ for P not simplicial?

C. Small g_2 P simplicial

$g_2(P) = 0 \overset{\text{Kalai}}{\iff} P$ is stacked

Def: P is prime if it has no missing $d-1$ face.



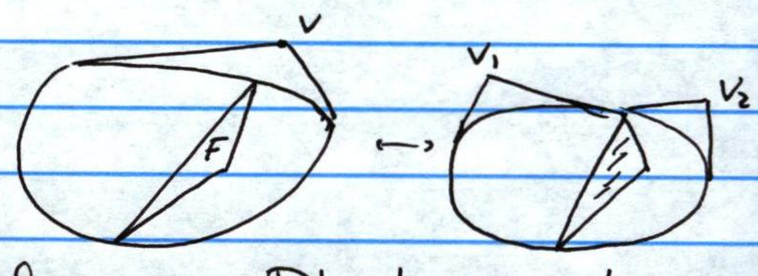
$g_2(P_1 \# P_2) = g_2(P_1) + g_2(P_2)$.

For P prime

$g_2 = 1 \underset{\text{N.-Novinski}}{\iff} P = \partial(\text{gon} \oplus \sigma^{d-2})$

or $= \partial(\sigma^i \oplus \sigma^{d-i})$

$g_2 = 2 \underset{\text{Zheng}}{\iff}$ Char. based on Swartz operation:



Swartz: There are only finitely many PL-types s.t. $g_2(M) \leq b$, b constant.

D. large g_2 (P simplicial)

Thm (Adiprasito - N - Samper):

① (Conj by Kalai '94)

$K = C^1$ convex body in $\mathbb{R}^d$, $d \geq 4$

$P_m \xrightarrow[m\to\infty]{\text{Hausdorf}} K \implies g_2(P_m) \xrightarrow[m\to\infty]{} \infty$.

② $K = C^2$ convex body

$\implies$ for P close enough to K

$g_2(P) = \Omega\left(\delta^H(P,K)^{-\frac{d-1}{2}}\right)$.

E. Combinatorial classes of simplicial polytopes

1. balanced polytopes:

$$\frac{h_i}{\binom{d}{i}} \geq \frac{h_{i-1}}{\binom{d}{i-1}}$$

h-vector: $\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}$

Dehn-Somerville eq: $h_i = h_{d-i}$

Klee-Novik : $i = 2$

Junke-Kubitzke-Murai general case

2. centrally-symmetric polytopes: $x \in P \iff -x \in P$

Stanley '87 : $g_2 \geq \binom{d}{2} - \binom{d}{1}$.

Klee-N-Novik-Zheng: $) = \iff$ d-dim crosspolytope + symmetric stacking

3. flag polytopes : $\sum_0^d h_i x^i = \sum_0^{\lfloor \frac{d}{2} \rfloor} \gamma_i x^i (x+1)^{d-2i}$

Gal's conj : $\gamma_i \geq 0$

$\gamma_2 \geq 0$ $\left(\xRightarrow{\text{Flag-HPW}} f(P) \geq f(\Sigma \Sigma \ldots (\text{gon})) \right)$