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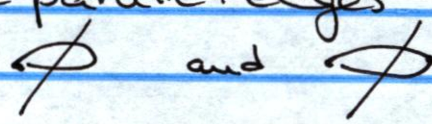
Generic toric varieties
& One sided pairings

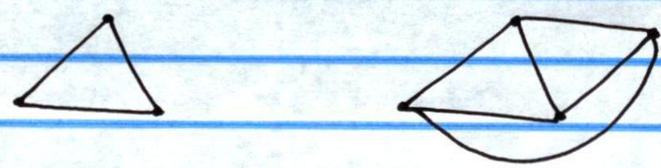
Karim Adiprasito

G planar graph ($G \hookrightarrow \mathbb{R}^2$)

Q: How many edges E can G have in terms of number of vertices V ?

Forbid loops and parallel edges
(G simple)





Descartes: $E \leq 3V-6$.

Proof: Start from G ; add edges until the complement of G consists of disjoint triangles. I have only added edges, so E has increased.

$$2E = 3T$$
$$V - E + T = 2$$
$$\Rightarrow \qquad E \leq 3V-6 .$$

Q: 2 dim simplicial complex $\xhookrightarrow{PL} \mathbb{R}^4$.
How large can $T = \#$ of triangles be in terms of $V = \#$ vertices and $E = \#$ edges?

Known: $T \leq V \cdot E$

$T \leq E(V^{8/9})$

Thm: $T < 4E$ (sharp bound with the proper constants, $T \leq 4E -$ binomial coeff)

Back to the planar graph:

$G \hookrightarrow S^2$ facewise linear,

i.e. can be completed to a triangulation Δ of S^2

$\rightarrow$ simplicial fan Σ over Δ.

To this I can associate a toric variety X_Σ

$H^*(X_\Sigma)$ is a Poincaré duality algebra with fundamental class in $H^6(X_\Sigma)$.

H depends on combinatorics of Δ
+ positions of rays of Σ (vertices of Δ)

$\underset{(d-1)\text{-sphere}}{\Delta} \rightsquigarrow \mathbb{R}[\Delta] := \mathbb{R}[x] / \langle x^\alpha : \operatorname{supp}\alpha \notin \Delta \rangle$

Artinian reduction $A(\Delta) = \mathbb{R}[\Delta] / \langle \Theta_1, \dots, \Theta_d \rangle$

$A(\Delta)$ is graded

$A^i(\Delta)$ is generated by $(i-1)$-dim faces of Δ.

$\dim A^i(\Delta) \leq f_{i-1}(\Delta)$.

$\Gamma \subseteq \Delta$, $A(\Gamma) = A(\Delta) / \langle x^\alpha : \operatorname{supp}\alpha \notin \Gamma \rangle$

and $\dim A^i(\Gamma) \leq f_{i-1}(\Gamma)$.

We are interested in graphs,

$$\dim A^1(\Gamma) \leq f_0(\Gamma).$$

Lower bounds for $\dim A^i(\Gamma)$:

$$\dim A^i(\Gamma) \geq f_{i-1}(\Gamma) - i f_{i-2}(\Gamma)$$

In the Descarte theorem setting:

$$\dim A^2(\Gamma) \geq f_1(\Gamma) - 2 f_0(\Gamma).$$

Want: $\dim A^1(\Gamma) \geq \dim A^2(\Gamma)$

$A(\Delta)$ is a Poincaré Duality algebra,

$$A^1(\Delta) \times A^2(\Delta) \longrightarrow A^3(\Delta) = \mathbb{R}$$

perfect $\Longrightarrow$ $\dim A^1(\Delta) = \dim A^2(\Delta)$.

$I(\Delta, \Gamma) = \ker : A(\Delta) \to A(\Gamma)$; we want
$\dim I^1(\Delta, \Gamma) \leq \dim I^2(\Delta, \Gamma)$.

How to prove this,

If $A^1(\Delta) \xrightarrow[\sim]{\cdot h} A^2(\Delta)$ for some $h \in A^1(\Delta)$,

$$\begin{tikzcd} A^1(\Delta) \arrow[r, "\cdot h", "\sim"'] & A^2(\Delta) \\ I^1(\Delta,\Gamma) \arrow[u] \arrow[r] & I^2(\Delta,\Gamma) \arrow[u] \end{tikzcd}$$

Restrict pairing to the ideals:

$$I^1(\Delta,\Gamma) \times I^2(\Delta,\Gamma) \longrightarrow I^3(\Delta,\Gamma) \cong A^3(\Delta) \cong \mathbb{R}$$

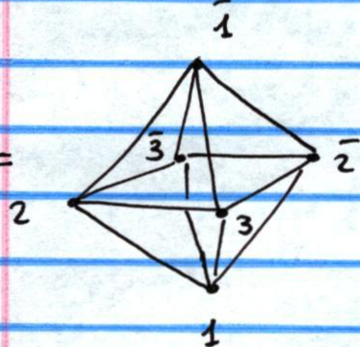
non degenerate on the left.

Not true in general, for any Artinian reduction.

Thm: For Δ 2-sphere, $\exists A(\Delta)$ Artinian reduction s.t. $\forall \Gamma \subseteq \Delta$ we have that

$$(*) \qquad I^1(\Delta,\Gamma) \times I^2(\Delta,\Gamma) \longrightarrow I^3(\Delta,\Gamma) \quad \text{is}$$

non degenerate on the left.

$\diamond$ = 

$$\mathbb{R}[\diamond] \big/ \langle x_{\bar{1}} - x_1,\ x_{\bar{2}} - x_2,\ x_{\bar{3}} - x_3 \rangle$$

Let Γ be the lower hemisphere.

$$I(\Delta,\Gamma) = \langle x_{\bar{1}} \rangle \qquad\qquad x_{\bar{1}} \cdot x_{\bar{1}} = 0$$

Here $(*)$ does not hold.

Thm: Δ 4-sphere. $\exists A(\Delta)$ such that $\forall \Gamma \subseteq \Delta$, $I^2(\Delta,\Gamma) \times I^3(\Delta,\Gamma) \longrightarrow \underset{\mathbb{R}^2}{\underset{\text{ns}}{I^5(\Delta,\Gamma)}}$ is non degenerate on the left.

Cor: $T \leq 4E$.