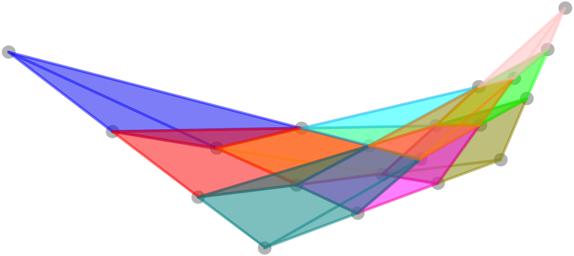
Combinatorics of the (tree) amplituhedron

Lauren K. Williams, UC Berkeley



Based on joint work with Steven Karp and Yan Zhang

Outline:

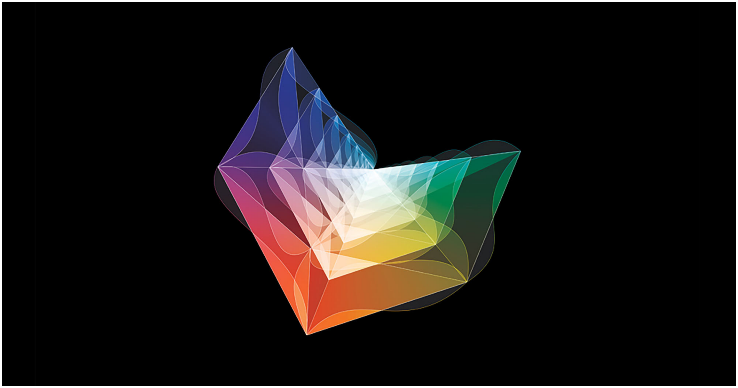
- Background + motivation
- What does the amplituhedron look like?
- Triangulating the amplituhedron
- Conjectures on numerology of $\mathcal{A}_{n,k,m}$

Background and Motivation for the amplituhedron

- Introduced by physicists Arkani-Hamed and Trnka in 2013

- Its "volume" is supposed to compute scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory

- A "jewel at the heart of quantum physics" – Wired Magazine.



- #10 among the top 100 top stories of 2013, Discover Magazine.

Background + Motivation on the Amplituhedron

- Despite the name, it's <u>not</u> a polyhedron
- But the amplituhedron does generalize:
  - polygons
  - cyclic polytopes
  - the positive Grassmannian
- Beautiful combinatorics...

The positive Grassmannian

Def: The Grassmannian $Gr_{k,n} = \{V \subset \mathbb{R}^n \mid \dim V = k\}$

Represent $V \in Gr_{k,n}$ by full rank $k \times n$ matrix $A = (A_1 | \ldots | A_n)$

For $J \in \binom{[n]}{k}$, $\Delta_J(A)$:= minor of A using columns J.
Plucker coordinate.

The totally non-negative Grassmannian is

$$Gr_{k,n}^{\geq 0} := \{A \in Gr_{k,n} \mid \Delta_J(A) \geq 0 \ \forall \ J \in \binom{[n]}{k}\}$$

The totally positive Grassmannian is

$$Gr_{k,n}^{>0} := \{A \in Gr_{k,n} \mid \Delta_J(A) > 0 \ \forall \ J \in \binom{[n]}{k}\}$$

<u>Def</u>: Given $\mathcal{M} \subseteq \binom{[n]}{k}$, set

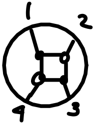
$$S_{\mathcal{M}} := \{ A \in Gr_{kn}^{\geq 0} \mid \Delta_J(A) > 0 \text{ iff } J \in \mathcal{M} \}.$$

"positroid cell."

<u>Thm (Postnikov)</u>: If $S_{\mathcal{M}} \neq \phi$ then $S_{\mathcal{M}}$ is open ball.

So have cell decomposition $Gr_{kn}^{\geq 0} = \coprod S_{\mathcal{M}}$

(Nonempty) cells in bijection with:
- decorated permutations
- (equivalence classes of) reduced plabic graphs, ie, on shell diagrams



- Le-diagrams

+	+	0	+	0	+
0	0	0	0	+	+
+	+	+	+		
0	0	0	+		

<u>Def</u>: A <u>⅃-diagram</u> for $Gr_{kn}^{\geq 0}$ is a Young diagram $\leq k \times (n-k)$ filled with $0, +$ s.t. no

$$\begin{array}{ccc} & & + \\ & & \vdots \\ + & \cdots & 0 \end{array}$$

<u>Ex:</u>

0	+	+	0
0	0	0	+
+	+	+	

<u>Thm (Postnikov)</u>: Cells of $Gr_{kn}^{\geq 0}$ in bijection with ⅃-diagrams.

From ⅃-diagram can read off all points of the cell (as matrices or in terms of Plucker coord's)

Dim of cell = # of +'s.

The Amplituhedron

Def: (Arkani-Hamed, Trnka) Let Z be a $(k+m) \times n$ real matrix w/ maximal minors positive.

$\rightsquigarrow$ map $\tilde{Z}: Gr_{kn}^{\geq 0} \longrightarrow Gr_{k,k+m}$ defined by:

If A a $k \times n$ matrix representing a point in $Gr_{kn}^{\geq 0}$,

$$A \longmapsto AZ^t = k\overset{k+m}{\left(\quad \right)}$$

$\uparrow$

Lemma: This has rank k.

Pf: Uses positivity of both A and Z.

<u>Def</u>: (Arkani-Hamed, Trnka) Let Z be a $(k+m)\times n$ real matrix w/ maximal minors positive. $k+m \leq n$

$\rightsquigarrow$ map $\tilde{Z}: Gr_{kn}^{\geq 0} \longrightarrow Gr_{k,k+m}$ defined by:

If A a $k\times n$ matrix representing a point in $Gr_{kn}^{\geq 0}$,

$$A \longmapsto AZ^t = {}_k\overset{k+m}{\left(\quad\right)}$$

Equiv, $\langle v_1, \ldots, v_k\rangle \longmapsto \langle Zv_1, \ldots, Zv_k\rangle$

k-dim'l subspace of $\mathbb{R}^n$ — k-dim'l subspace of $\mathbb{R}^{k+m}$

The <u>(tree)</u> <u>amplituhedron</u> $\mathcal{A}_{n,k,m}$ is $\tilde{Z}(Gr_{kn}^{\geq 0}) \subset Gr_{k,k+m}$

$\mathcal{A}_{n,k,m}$ generalizes many nice objects

(1) If Z a square matrix, i.e. $k+m=n$, then

$$\tilde{Z}: (Gr_{k,n})_{\geq 0} \longrightarrow Gr_{k,n}(\mathbb{R})$$

$$A \longmapsto AZ^{t} \quad \text{is injective}$$

so $\mathcal{A}_{n,k,m}(Z) \cong (Gr_{k,n})_{\geq 0}$

$\mathcal{A}_{n,k,m}$ generalizes many nice objects

(2) If $k=1$, $m=2$, $\mathcal{A}_{n,k,m}(Z) \subset Gr_{1,3} = \mathbb{P}^2$ is a polygon in $\mathbb{P}^2$:

$$\underset{a_i \geq 0}{(a_1 : \dots : a_n)} \longmapsto (a_1 \dots a_n) \begin{array}{c} \\ {\scriptstyle n} \end{array} \begin{array}{c} {\scriptstyle 3} \\ \begin{pmatrix} \\ Z^t \\ \\ \end{pmatrix} \end{array} \subset \mathbb{P}^2$$

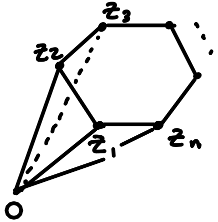
Let $e_i := (0 \dots \overset{i\text{th pos}}{1} \dots 0)$ and write $Z = {}_3\left(z_1 \middle| z_2 \middle| \dots \middle| z_n \right)$

Then $e_i \mapsto z_i$ (a point in $\mathbb{P}^2$)

Max minors of Z positive $\Rightarrow$ z_i's in convex position.

So as a_i's vary over $\mathbb{R}_{\geq 0}$,

$(a_1 \dots a_n) \begin{pmatrix} Z^t \end{pmatrix}$ gives all points in cone spanned by z_i's.



$\mathcal{A}_{n,k,m}$ generalizes many nice objects

(3) If $k=1$, $\mathcal{A}_{n,k,m}(Z) \subset Gr_{1,m+1} = \mathbb{P}^m$

is combinatorially equiv. to cyclic polytope

with n vertices in $\mathbb{P}^m$.

(Distinguished among simplicial polytopes for maximizing # faces, given dim + # vertices)

Embarrassing open question

- To what extent does $\mathcal{A}_{n,k,m}(Z)$ depend on the matrix Z?

- If $Z = Z_0$ s.t. rows of Z_0 span the unique element of $Gr_{Z_0}(k+m, n)$ fixed by cyclic action, then $\mathcal{A}_{n,k,m}(Z_0)$ is a closed ball (Galashin-Karp-Lam).

- If $m=1$ or $k=1$, $\mathcal{A}_{n,k,m}(Z)$ always a ball ...

- Combinatorics & topology should not depend on Z, but this is unknown ...

Triangulating the amplituhedron

Recall $\mathcal{A}_{n,k,m}(Z) = \tilde{Z}(Gr_{kn}^{\geq 0})$ where $\tilde{Z}: Gr_{kn}^{\geq 0} \longrightarrow Gr_{k,k+m}$.

Image has full dimension km

$m=4$ most interesting for physics.

<u>Conj (AH-T)</u>: There is a "BCFW" collection of $4k$-dim'l cells in $Gr_{kn}^{\geq 0}$ whose images "triangulate" $\mathcal{A}_{n,k,4}$ — ie. images are disjoint & cover a dense subset of $\mathcal{A}_{n,k,4}$.

<u>Note</u>: # of cells in this collection is Narayana number

$$N_{k+1,n-3} = \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}.$$

Physical significance of conj: implies one can calculate scattering amplitudes by integrating form over amplituhedron.

(Previous method: add up multiple contributions, one for each BCFW cell)

Note: Conj. does not really depend on Z

Conjecture true for $k=1$: reduces to result of Rambau on triangulations of cyclic polytopes.

Conjecture hard for $m=4$!
Let's start by looking at $m=1$...

Orthogonal point of view on $\mathcal{A}_{n,k,m}$

- $\mathcal{A}_{n,k,m} \in Gr_{k,k+m}$. For small m, prefer to work with $Gr_{m,k+m} \cong Gr_{k,k+m}$.

<u>Theorem</u> (Karp, W.): Let $Z \in Mat^{>0}_{k+m,n}$.
Let $W \subset \mathbb{R}^n$ be rowspan(Z).
Let $\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in Gr^{\geq 0}_{k,n}\} \subseteq Gr_m(W)$.

Then $\mathcal{A}_{n,k,m}(Z)$ homeomorphic to $\mathcal{B}_{n,k,m}(Z)$.

<u>Pf idea:</u>

$$\underbrace{V^\perp \cap W}_{\substack{m\text{-dim'l} \\ \text{subspace} \\ \text{of } \mathbb{R}^n}} \xmapsto[\substack{\text{Take} \\ \text{orthog} \\ \text{complement} \\ \text{in } \mathbb{R}^n}]{} V + W^\perp \overset{Z:\mathbb{R}^n \to \mathbb{R}^{k+m}}{\longmapsto} \underbrace{Z(V)}_{\substack{k\text{-dim'l} \\ \text{subspace} \\ \text{of } \mathbb{R}^{k+m}}} \subseteq Gr_{k,k+m}$$

Note: $Z(W^\perp) = \{0\}$

Sign Variation

Def: For $v \in \mathbb{R}^n$, let $\mathrm{var}(v) = \#$ times v changes sign, reading coordinates L to R

e.g. for $v = (4,-1,0,-2)$, $\mathrm{var}(v) = 1$.

Let $\overline{\mathrm{var}}(v) =$ max # sign changes after we choose a sign for each 0 coordinate.

e.g. $\overline{\mathrm{var}}(4,-1,0,-2) = 3$.

Theorem (Gantmacher-Krein, 1950): Let $V \in Gr_{k,n}(\mathbb{R})$.

(i) $V \in Gr_{kn}^{\geq 0} \iff \forall$ vectors $x \in V$, $\mathrm{var}(x) \leq k-1$
$\iff \forall$ vectors $w \in V^\perp$, $\overline{\mathrm{var}}(w) \geq k$

(ii) $V \in Gr_{kn}^{>0} \iff \forall$ vectors $x \in V \setminus \{0\}$, $\overline{\mathrm{var}}(x) \leq k-1$
$\iff \forall$ vectors $w \in V^\perp \setminus \{0\}$, $\mathrm{var}(w) \geq k$

Ex: For $a, b, c, d \geq 0$ and $bc - ad \geq 0$,

$A = \begin{pmatrix} 1 & 0 & -a & -b \\ 0 & 1 & c & d \end{pmatrix}$ represents point of $Gr_{24}^{\geq 0}$.

GK: $V \in Gr_{kn}^{\geq 0} \iff \forall$ vectors $x \in V$, $\mathrm{var}(x) \leq k-1$

Check that row vectors of A (+ linear combo's) satisfy $\mathrm{var}(x) \leq k-1 = 1$.

Simple Description of Amplituhedron

Theorem (Karp, W.): For $W \in Gr_{k+m,n}^{>0}$, we have:

① $\mathcal{B}_{n,k,m}(W) \subseteq \{X \in Gr_m(W) \mid k \leq \overline{\text{var}}(x) \leq k+m-1 \ \forall x \in X\} \subseteq Gr_m(W)$

Moreover, when $m=1$,

② $\mathcal{B}_{n,k,1}(W) = \{x \in \mathbb{P}(W) \mid \overline{\text{var}}(x) = k\}$

Open: In ①, is the $\subseteq$ an $=$? True for $m=1$, $k+m=n$.

Questions: ① Can we triangulate $\mathcal{A}_{n,k,1}$? ② What does it look like?

$m=1$ Amplituhedron

Recall: $\mathcal{B}_{n,k,1}(W) = \{ x \in \mathbb{P}(W) \mid \overline{\mathrm{var}}(x) = k \}$

Def: Let $\overline{\mathrm{Sign}_{n,k,1}} \subseteq \{0,+,-\}^n$ be the set of sign vectors σ s.t. $\overline{\mathrm{var}}(\sigma) = k$.

Let $\mathrm{Sign}_{n,k,1}$ " " s.t. $\mathrm{var}(\sigma) = k$.

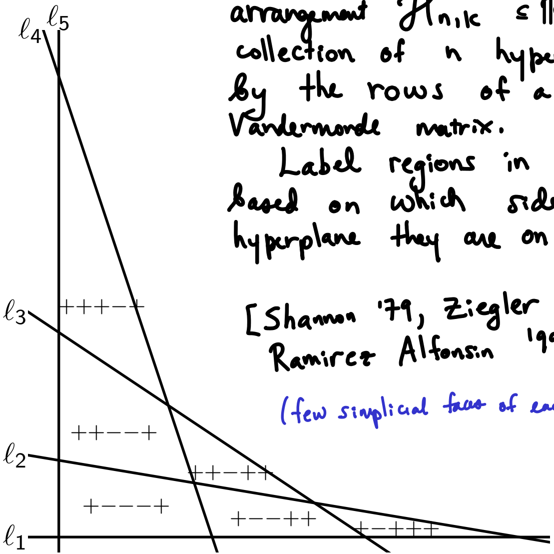
E.g. for $n=5$, $k=2$,

$\mathrm{Sign}_{5,2,1} = \{ +++-+,\ ++--+,\ ++-++,\ +---+,\ +--++,\ +-+++, \dots \}$

Let $\mathcal{B}_\sigma(W) := \{ x \in \mathcal{B}_{n,k,1}(W) \mid \mathrm{sign}(x) = \sigma \}$

So $\mathcal{B}_{n,k,1}(W) = \coprod_{\sigma \in \mathrm{Sign}_{n,k,1}} \mathcal{B}_\sigma(W)$.

Interpret $\mathrm{Sign}_{5,2,1}$ as regions of arrangement?



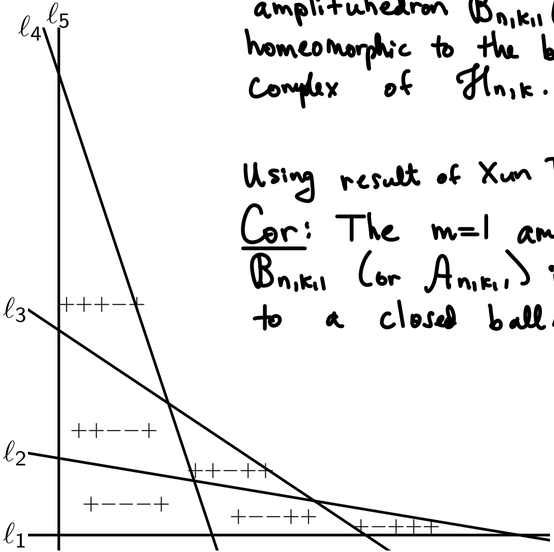
Def: The cyclic hyperplane arrangement $\mathcal{H}_{n,k} \subseteq \mathbb{R}^k$ is the collection of n hyperplanes given by the rows of a $n \times (k+1)$ Vandermonde matrix.

Label regions in complement based on which side of each hyperplane they are on.

[Shannon '79, Ziegler '93, Ramirez Alfonsin '99, w/Forge '01]

(few simplicial faces of each dim)

The cyclic hyperplane arrangement and $\mathcal{A}_{5,2,1}$



<u>Thm (Karp, W)</u>: The amplituhedron $\mathcal{B}_{n,k,1}(W)$ is homeomorphic to the bounded complex of $\mathcal{H}_{n,k}$.

Using result of Xun Dong,

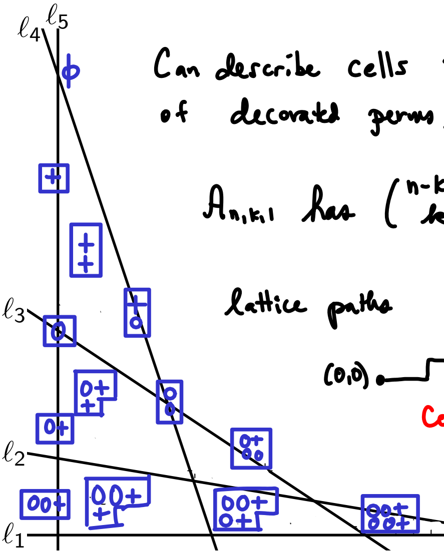
<u>Cor</u>: The $m=1$ amplituhedron $\mathcal{B}_{n,k,1}$ (or $\mathcal{A}_{n,k,1}$) is homeomorphic to a closed ball.

The cyclic hyperplane arrangement and $\mathcal{A}_{5,2,1}$

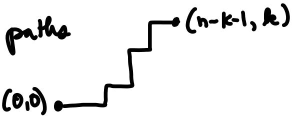
Cell decomp: $\mathcal{B}_{n,k,1}(W) = \coprod_{\sigma \in \mathrm{Sign}_{n,k,1}} \mathcal{B}_\sigma(W)$,

Can describe cells in terms of decorated perms, or ⅃-diagrams...

$\mathcal{A}_{n,k,1}$ has $\binom{n-k-1}{k}$ cells $\longleftrightarrow$

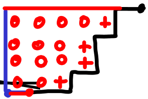


lattice paths



$\longleftrightarrow$

Corresponding ⅃-diagram

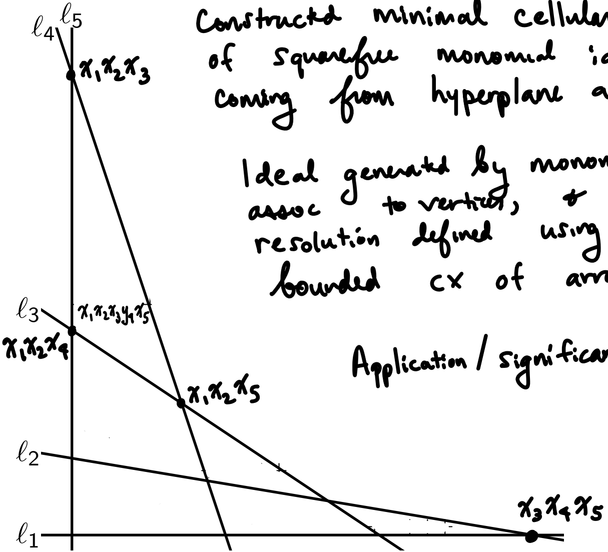


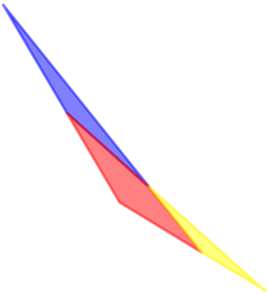
<u>Aside</u>: Novik–Postnikov–Sturmfels 2000

Constructed minimal cellular resolutions of squarefree monomial ideals coming from hyperplane arrangements.

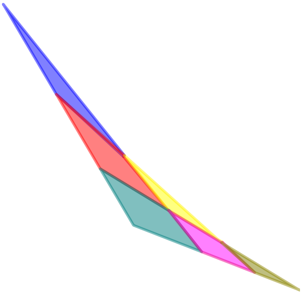
Ideal generated by monomials assoc to vertices, & resolution defined using the bounded cx of arrangement.

Application / significance?

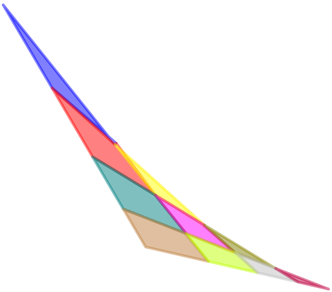




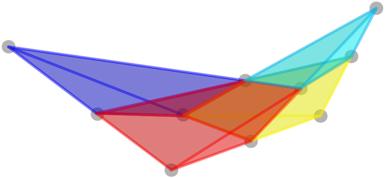
$\mathcal{A}_{4,2,1}$



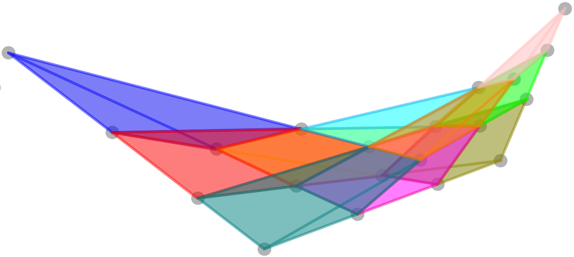
$\mathcal{A}_{5,2,1}$



$\mathcal{A}_{6,2,1}$



$\mathcal{A}_{5,3,1}$



$\mathcal{A}_{6,3,1}$

Next: $m = 4$

Recall

<u>Conj (AH – T)</u>: The BCFW cells (which have dim $4k$) in $Gr_{kn}^{\geq 0}$ give a "triangulation" of $\mathcal{A}_{n,k,4}$: ie.

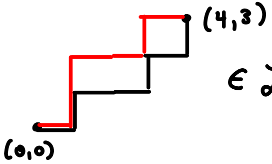
ie. their images are disjoint & cover a dense subset of $\mathcal{A}_{n,k,4}$.

What are the BCFW cells?
Physicists described them via very complicated recurrence.
We give <u>explicit</u> description...

Recall: # of cells in conjectural triangulation of $\mathcal{A}_{n,k,4}$ is Narayana number $N_{k+1,n-3} = \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$.

Let $\mathcal{L}_{n,k,4}$ = set of all pairs of non-crossing lattice paths taking steps W and S from $(n-k-4, k)$ to $(0,0)$.

In particular, $N_{k+1,n-3} = |\mathcal{L}_{n,k,4}|$.

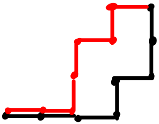


$\in \mathcal{L}_{11,3,4}$

Theorem (Karp, W., Zhang): Explicit description of all BCFW cells for $\mathcal{A}_{n,k,4}$. Give bijection

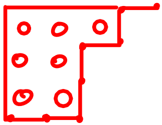
$$\mathcal{L}_{n,k,4} \longrightarrow \text{⅃-diagrams of BCFW cells}$$

Ex:



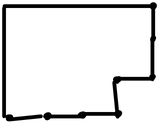
$\in \mathcal{L}_{n,k,4}$. $\rightsquigarrow$

Step 0:



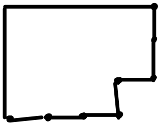
Y_u

and



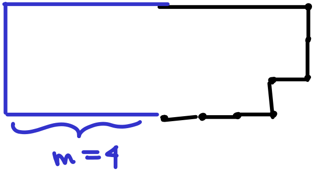
Y_L

Step 1: Use Y_L to get shape of ⅃-diagram

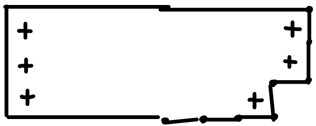


Y_L

$\longmapsto$

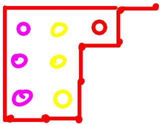


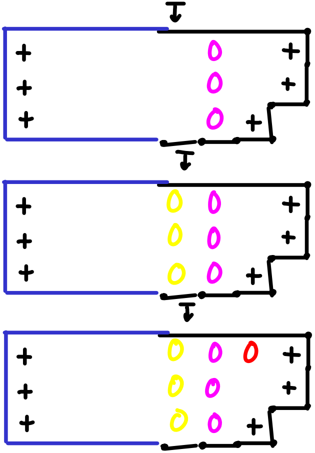
Step 2: Put + at L and R of each row



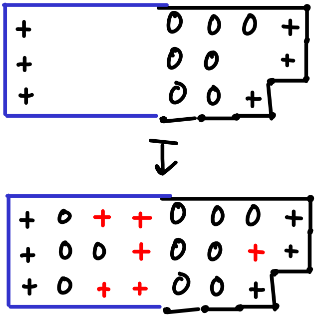
Step 3: Place columns of 0's in Y_u L to R into diagram.

Recall

$Y_u =$ 



Step 4: Place 2 more +'s in each row, justified to right



Step 5: Fix blocked 0's with J-move $\begin{matrix}+ & +\\ + & 0\end{matrix} \mapsto \begin{matrix}0 & +\\ + & +\end{matrix}$

Alternatively, can read off perm from this diagram (simple algorithm).

Theorem (Karp, W., Zhang): Explicit description of all BCFW cells for $\mathcal{A}_{n,k,4}$. Give bijection

$$\mathcal{L}_{n,k,4} \longrightarrow \text{Le-diagrams of BCFW cells}$$

Rk: Can also give bijection
Dyck paths $\longrightarrow$ BCFW cells (bases) (Karp, Thomas, W, Zhang)

We then use sign variation techniques to prove

Theorem (KWZ): When $k=2$, the images of the BCFW cells are disjoint in $\mathcal{A}_{n,k,4}$.

Recall: $\mathcal{A}_{n,k,4} = \tilde{Z}(Gr_{kn}^{\geq 0})$, where

$\tilde{Z}: Gr_{kn}^{\geq 0} \to Gr_{k,k+4}$ defined by $A \mapsto {}_k\Big(\overset{n}{\quad}\Big)_n\Big(\overset{k+4}{\quad}\Big)$

Theorem (Karp-W.-Zhang): Let $k=2$. The images of two distinct BCFW cells in $\mathcal{A}_{n,k,4}$ are distinct.

Idea: Suppose V_1 and $V_2 \in Gr_{kn}^{\geq 0}$ lie in 2 BCFW cells.

① $V_i \in Gr_{kn}^{\geq 0} \overset{\text{G.K.}}{\Longrightarrow} \forall x \in V_i$, $\text{var}(x) \leq k-1$. Variation small.

② If $\tilde{Z}(V_1) = \tilde{Z}(V_2)$, ie. $V_1 - V_2 \in \ker(Z)$, then

$Z \in Gr_{k+4,n}^{>0} \overset{\text{G.K.}}{\Longrightarrow} \forall$ nonzero $y \in V_1 - V_2$, $\text{var}(y) \geq k+4$. Variation big

Next: The BCFW cells (which supposedly triangulate the amplituhedron $\mathcal{A}_{n,k,4}$) only make sense for $m=4$.

What could we say for general m?

Conjectures on numerology of $\mathcal{A}_{n,k,m}$

	# max'l cells in decomp. of $\mathcal{A}_{n,k,m}$	
$m=1$	$\binom{n-1}{k}$	Karp-W. (theorem)
$m=2$	$\binom{n-2}{k}$	Arkani-Hamed-Trnka-Thomas
$m=4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	Conj of AH-T
$k=1$	$\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$	$\mathcal{A} \cong$ cyclic polytope $C(n,m)$

Is there a formula which generalizes all of these?

Conjectures on numerology of $\mathcal{A}_{n,k,m}$

Let $N(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$

Note: $N(a,b,c)$ symmetric in a,b,c.

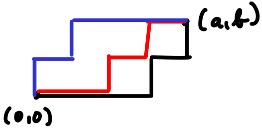
Conj: KWZ (90% confidence?) For even m, there is cell decomposition of $\mathcal{A}_{n,k,m}$, coming from images of cells of $Gr_{kn}^{\geq 0}$, which has $N(k, n-m-k, \frac{m}{2})$ top-dim'l cells (of dim. km)

Conj: KWZ (50% confidence??) For odd m, " " $\mathcal{A}_{n,k,m}$ " " which has $N(k, n-m-k, \frac{m+1}{2})$ top-dim'l cells.

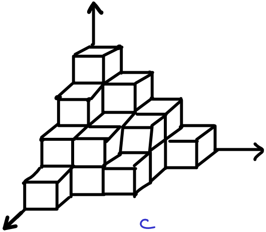
Rk: These conjectures generalize all previous results/ conjectures.

$N(a,b,c)$ counts:

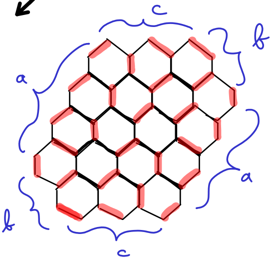
- collections of c noncrossing paths from (a,b) to $(0,0)$ taking steps W and S



- plane partitions $\leq a \times b \times c$ box



- Kekulé structures (perfect matchings) of a hexagon-shaped benzenoid w/ parameters a,b,c)



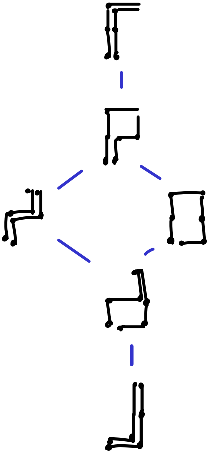
So conjecturally there's a triangulation of $A_{n,k,m}$ whose top-dim'l cells are in bijection with:

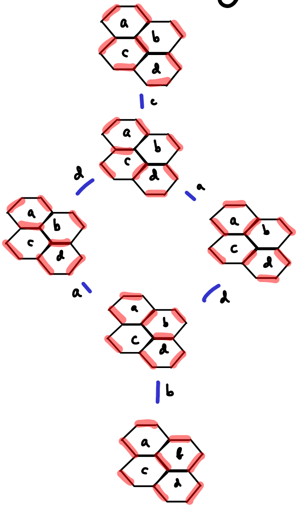
- collections of $\lfloor \frac{m+1}{2} \rfloor$ noncrossing paths from $(k, n-m-k)$ to $(0,0)$ taking steps W and S

- plane partitions $\leq k \times (n-m-k) \times \lfloor \frac{m+1}{2} \rfloor$ box

- Kekulé structures (perfect matchings) of a hexagon-shaped benzenoid w/ parameters $k, n-m-k, \lfloor \frac{m+1}{2} \rfloor$

<u>Note:</u> These combinatorial objects all have structure of distributive lattice (= really nice kind of partially ordered set)

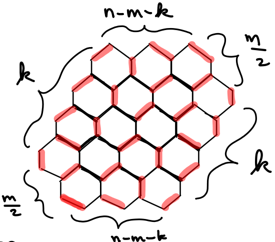




(Containment of plane partitions)

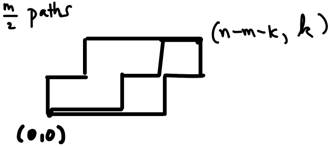
What does this mean for the amplituhedron?

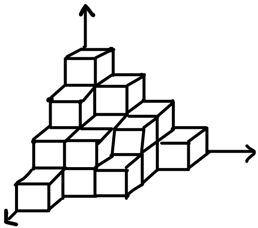
Thank you!



Part 1 of talk joint w/ Steven Karp,
"the $m=1$ amplituhedron + cyclic hyperplane arrangements,"
IMRN 2017

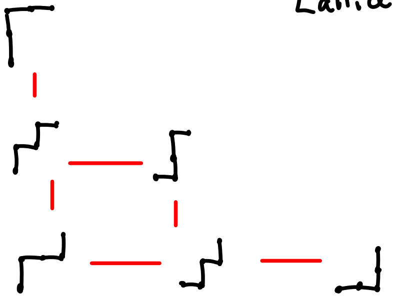
Part 2 of talk joint w/ Karp + Yan Zhang (appendix joint w/ Hugh Thomas)
"Decompositions of amplituhedra,"
arXiv:1708.09525





$m=1,\ n=5,\ k=2$

Adjacency of max'l cells in $A_{n,k,m}$ $\longleftrightarrow$ Lattice structure on paths



The 9 classes of BCFW cells for $k = 2$.

Class 1.

+	0	0	0	+	+	+
+	+	+	+			

+	+	+	+	0	0	0	0	-	o
0	0	0	0	+	+	+	+	+	

Class 2.

+	0	0	+	+	+
+	+	+	+		

+	+	+	+	0	0	0	-	o
0	0	0	+	+	+	+	+	

Class 3.

+	0	+	+	+
+	+	+	+	

+	+	+	+	0	0	-	o
0	0	+	+	+	+	+	

Class 4.

+	+	+	0	+
+	+	+	+	

+	+	0	0	-	-	-	d
+	+	+	+	+	+	0	

Class 5.

+	+	+	0	0	+
+	0	+	+	+	

+	+	0	0	0	-	-	-	d
+	+	+	+	+	+	0	0	

The 9 classes of BCFW cells for $k = 2$.

Class 6. $\begin{array}{|c|cc|cc|c|c|} + & + & + & 0 & 0 & 0 & + \\ \hline + & 0 & 0 & + & + & + & \end{array}$ $\begin{array}{|cc|cc|cc|cc|c} + & + & 0 & 0 & 0 & 0 & - & - & - \\ \hline + & + & + & + & + & + & 0 & 0 & 0 \end{array}$ d

Class 7. $\begin{array}{|c|cc|cc|c|} + & + & + & 0 & 0 & + \\ \hline + & 0 & 0 & + & + & + \end{array}$ $\begin{array}{|ccc|cc|cc|c} + & + & 0 & 0 & 0 & - & - & - \\ \hline + & + & + & + & + & 0 & 0 & 0 \end{array}$ d

Class 8. $\begin{array}{|c|ccc|c|} + & + & + & 0 & + \\ \hline + & 0 & + & + & + \end{array}$ $\begin{array}{|ccc|ccc|c} + & + & 0 & 0 & - & - & - \\ \hline + & + & + & + & + & 0 & 0 \end{array}$ d

Class 9. $\begin{array}{|c|cc|c|} + & + & + & + \\ \hline + & + & + & + \end{array}$ $\begin{array}{|ccc|cc|c} + & + & 0 & - & - & - \\ \hline + & + & + & + & + & 0 \end{array}$ d