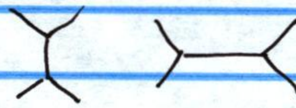
Unwinding the Amplituhedron — Hugh Thomas

joint with Nima Arkani-Hamed, Jaroslav Trnka

1. History $N=4$ symmetric planar limit
studying the integrand

- Feynman diagrams '48



- Britto Cachazo Feng Witten '05 recursive approach to calculation

  individual terms not meaningful

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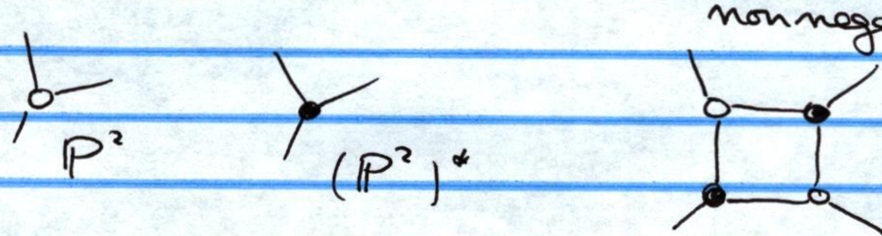
- Postnikov '06 math:0601764

- Arkani-Hamed Bourjaily Chachazo Goncharov Postnikov Trnka 1212.5605

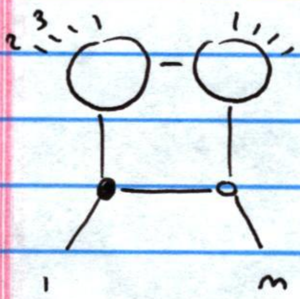
  terms in recursion
  $\updownarrow$
  BCFW cells in totally nonnegative Grassmannian



$\mathbb{P}^2$ $(\mathbb{P}^2)^*$

$Gr_{\geq}(1,3)$ $Gr_{\geq}(2,3)$ "on shell diagrams" $Gr_{\geq 0}(2,4)$

BCFW recursion



cells of high codim in $Gr_{\geq 0}(k,n)$, and they don't fit together.

Arkani-Hamed
Trnka

1312.2007

2. Amplituhedron

n particles

$0 \leq k \leq n$ helicities

ℓ loop level ($\ell = 0$)

m $= 4$ is physically meaningful

$= 2$ is what the talk focuses on.

$Z_1, \dots, Z_n \in \mathbb{R}^{k+m}$ particle momenta

$$Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$$

$n \geq k+m$ with maximal minors positive.

$$Gr_{\geq 0}(k,m) = GL_k(\mathbb{R}) \Big\backslash \left\{ X \in Mat_{k\times m} \;\; \begin{array}{l}\text{all maximal} \\ \text{minors of the} \\ \text{same sign}\end{array} \right\}$$

$$\mu_Z : Gr_{\geq 0}(k,m) \longrightarrow Gr(k, k+m)$$
$$C \longrightarrow CZ$$

$$\mathcal{A} = \mathcal{A}(m, k, m, \ell=0, Z) = \mu_Z(Gr_{\geq 0}(k,m)).$$

Conj (AH-T): C_i BCFW cells
$\mu_Z(C_i)$ subdivide $\mathcal{A}$

3. Special case, $k=1$

$C \in Gr_{\geq 0}(1, m)$ so $C = (c_1, \dots, c_m)$ where $c_i \in \mathbb{R}_{>0}$.

$\mathcal{A} \subseteq Gr(1, 1+m) = \mathbb{P}^m \ni Z_i$

multiplying by C $\rightsquigarrow$ taking convex combinations

By the assumptions on Z, $\mathcal{A}$ is a cyclic polytope with vertices Z_i.

simplex $\rightsquigarrow$ $Gr_{\geq 0}(k,n)$ Lam 1506.00603
polytope $\rightsquigarrow$ $\mathcal{A}$ "Grassmanian polytopes".

4. Boundaries of $\mathcal{A}$

$$Y \in \mathcal{A} \Longrightarrow \det \left[\begin{array}{c} Y \\ \hline Z_{i_1} \\ Z_{i_1+1} \\ Z_{i_2} \\ Z_{i_2+1} \\ \vdots \end{array}\right] \begin{array}{l} \}\, k \\ \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} m \end{array} > 0 \qquad (*)$$

For $k=1$, this characterizes $\mathcal{A}$

5. In general, boundaries are insufficient:

$n=4, \; k=2, \; m=2, \quad Z = I_4$.

$\Rightarrow \mathcal{A} = Gr_{\geq 0}(2,4)$

$$Gr(2,4) = GL_2 \setminus \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

$$Y_{ij} = \det \begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix}$$

$$\mathcal{A} = y_{ij} > 0 .$$

$$y_{13}\, y_{24} = y_{12}\, y_{34} + y_{14}\, y_{23}$$

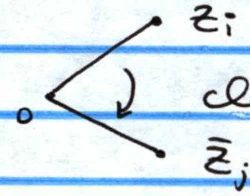
$$(*) \longrightarrow \quad y_{12}, y_{13} \; \cdots \; > 0$$

6. Topological description, $m=2$.

Given $Y \in Gr(k, k+2)$, is it in $\mathcal{A}$?

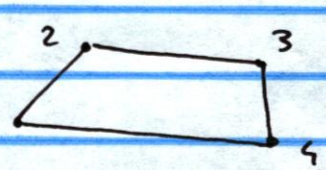
$Z_i \in \mathbb{R}^{k+2}$

$\bar{Z}_i \in \mathbb{R}^{k+2}/Y = \mathbb{R}^2$

$\det \begin{pmatrix} Y \\ Z_i \\ Z_j \end{pmatrix} > 0 \iff$ 

clockwise

(*) is satisfied for



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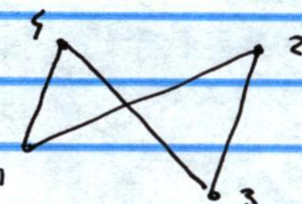
$Y \notin \mathcal{A}$



✓

$Y \in \mathcal{A}$

Thm: $\mathcal{A}_{m=2} = \left\{ Y \;\middle|\; \begin{array}{l} Y \text{ satisfies } (*) \text{ and} \\ \bar{Z}_1, \bar{Z}_2 \dots \bar{Z}_n \bar{Z}_1 \text{ winds} \\ \underbrace{\left\lfloor \tfrac{k+1}{2} \right\rfloor \text{ times}}_{\text{" maximally}} \text{ around the origin} \end{array} \right\}$