Reciprocal linear spaces, hyperbolicity, and determinants

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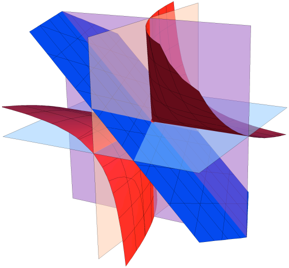

joint work with Mario Kummer, Max Planck Institute

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The plan

- Reciprocal linear spaces, hyperplane arrangements
- Multivariate real-rootedness and determinants
- Determinantal representations of reciprocal linear spaces
- A connection with graphic and simplicial matroids

Let $\mathcal{L}$ be a d-dim'l linear space in $\mathbb{C}^n$. Its reciprocal linear space is

$$\mathcal{L}^{-1} = \overline{\left\{\left(x_1^{-1}, \ldots, x_n^{-1}\right) : x \in \mathcal{L} \cap (\mathbb{C}^*)^n\right\}}.$$

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Proudfoot-Webster: $\mathbb{C}[\mathcal{L}^{-1}]$ is the intersection cohomology ring of the complement of a hyperplane arrangement.

Proudfoot-Speyer: Give flat degeneration of $\mathbb{C}[\mathcal{L}^{-1}]$ to Stanley-Reisner ring of broken circuit complex.

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Proudfoot-Speyer: Give flat degeneration of $\mathbb{C}[\mathcal{L}^{-1}]$ to Stanley-Reisner ring of broken circuit complex.

Varchenko: Critical points of products of linear forms are all real.

De Loera-Sturmfels-V: $\mathcal{L}^{-1} \cap (\mathcal{L}^{\perp} + v)$ are analytic centers of the bounded regions in a hyperplane arrangement.

Example: $n = 5, d = 3$

$$\mathcal{L} = \text{rowspan} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

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circuits $\{124, 135, 2345\}$

broken circuits $\{12, 13, 234\}$

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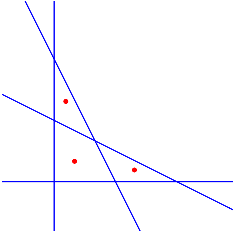
So $\mathcal{L}^{-1}$ has degree $|\mathcal{F}| = 4$.

Example: $(d, n) = (2, 4)$

Take $\ell_0, \ell_1, \ell_2, \ell_3 \in \mathbb{R}[s,t]_1$.

Then $\mathcal{L} = \{(\ell_0, \ell_1, \ell_2, \ell_3) : (s,t) \in \mathbb{R}^2\} \in \mathrm{Gr}(2,4)$



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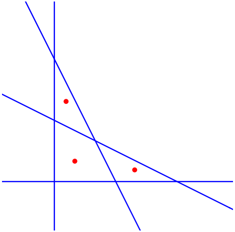
$\mathcal{L}$ intersects coord. hyperplanes $\cup_i \{x_i = 0\}$ in 4 points (proj.)
Remove them and take inverses to get

$$\mathcal{L}^{-1} = \overline{\{[\tfrac{1}{\ell_0}, \tfrac{1}{\ell_1}, \tfrac{1}{\ell_2}, \tfrac{1}{\ell_3}]\}} = \overline{\{[\ell_1\ell_2\ell_3, \ell_0\ell_2\ell_3, \ell_0\ell_1\ell_3, \ell_0\ell_1\ell_2]\}}.$$



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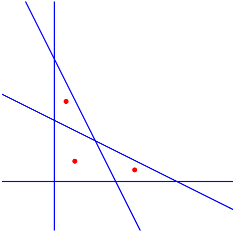
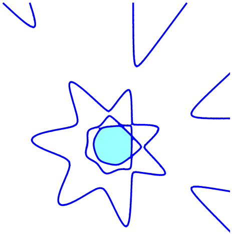


$\mathcal{L}^{-1}$ is a rational cubic curve.

For $v \in \mathbb{R}^4$, $\mathcal{L}^\perp + v$ intersects $\mathcal{L}^{-1}$ in $3 = \deg(\mathcal{L}^{-1})$ real points.

A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is hyperbolic w.r.t a point v if $f(v) \neq 0$ and every real line through v meets $\mathcal{V}_{\mathbb{C}}(f)$ in only real points.



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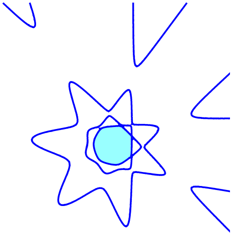

f is stable if it is hyperbolic w.r.t. all $v \in (\mathbb{R}_+)^n$.

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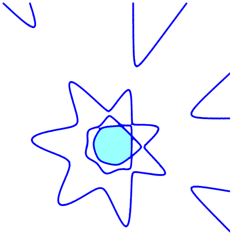

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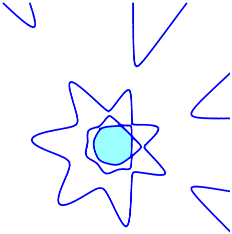

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Helton Vinnikov: For $n = 3$, every hyperbolic/stable polynomial $f \in \mathbb{R}[x_1, x_2, x_3]_d$ has such a determinantal representation.

Chow forms: making varieties into hypersurfaces since 1937

Let $X \subset \mathbb{P}^{n-1}$ be an irreducible variety of dimension $d-1$. Then

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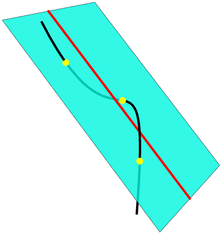
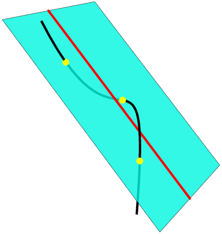
The Chow form of X is the resultant of these polynomials.

A real variety $X \subset \mathbb{P}^{n-1}(\mathbb{C})$ of $codim(X) = c$ is **hyperbolic** with respect to a linear space L of dim $c - 1$ if $X \cap L = \emptyset$ and for all real linear spaces $L' \supset L$ of $\dim(L') = c$, all points $X \cap L'$ are real.





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Theorem (Shamovich-Vinnikov 2015). If a curve $X \subset \mathbb{P}^{n-1}$ is hyperbolic with respect to L, then its Chow form is a determinant

$$\det\left(\sum_{I \in \binom{[n]}{2}} p_I(M) A_I\right) \quad \text{with} \quad \sum_{I \in \binom{[n]}{2}} p_I(L^\perp) A_I \succ 0$$

for some matrices $A_I \in \mathbb{R}^{D \times D}_{\text{sym}}$ with $D = \deg(X)$.

Reciprocal linear spaces are hyperbolic

Reformulated Varchenko: $\mathcal{L}^{-1}$ is hyperbolic with respect to $\mathcal{L}^{\perp}$.

Actually, $\mathcal{L}^{-1}$ is hyperbolic w.r.t. any linear space in $\mathrm{Gr}(n-d, n)$ whose Plücker coordinates agree in sign with those go $\mathcal{L}^{\perp}$.

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For $\sigma \in \{\pm 1\}^{\binom{n}{c}}$, let

$$\mathrm{Gr}(c,n)^{\sigma} = \{M \in \mathrm{Gr}(c,n) : \sigma_I \sigma_J p_I(M) p_J(M) \geq 0 \text{ for all } I, J\}.$$

A projective variety X of codimension c is σ-stable if it is hyperbolic with respect to all M in $\mathrm{Gr}(c,n)^{\sigma}$.

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Cor: If $\mathcal{L}^{\perp}$ is in the closure of $\mathrm{Gr}(n-d,n)^{\sigma}$, then $\mathcal{L}^{-1}$ is σ-stable.

Determinantal representation for $\mathcal{L}^{-1}$

Let $\mathcal{L} \in \mathrm{Gr}(d, n)$ not contained in a hyperplane $\{x_i = 0\}$.

Define $p(\mathcal{L}) \in \mathbb{P}(\bigwedge^d \mathbb{R}^n)$ and $\mathcal{B} = \{I \in \binom{[n]}{d} : p_I(\mathcal{L}) \neq 0\}$.

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Thm. The Chow form of $\mathcal{L}^{-1}$ can be written as a determinant

$$\det\left(\sum_{I \in \mathcal{B}} \frac{p_I(M)}{p_I(\mathcal{L})} A_I \right)$$

for some rank-one, p.s.d. matrices $A_I = v_I v_I^T$ of size $\deg(\mathcal{L}^{-1})$.

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Observations:

- Determinant is multiaffine (i.e. has degree ≤ 1 in $p_I(M)$).
- Matrix is positive definite when $\mathrm{sign}(p(M)) = \mathrm{sign}(p(\mathcal{L}))$.

A few words about the proof

Useful to consider $\mathbf{1} = (1, \ldots, 1) \in \mathcal{L}^{-1} \cap M^{\perp}$.

Let $\mathcal{H}_{\mathcal{B}} \subset \bigwedge^d \mathbb{R}^n$ denote the vector space

$$\mathcal{H}_{\mathcal{B}} \;=\; \mathrm{span}\left\{\gamma \wedge \mathbf{1} \;:\; \gamma \in \bigwedge^{d-1} \mathbb{R}^n\right\} \;\cap\; \mathrm{span}\{e_I : I \in \mathcal{B}\}$$

where $e_I = \wedge_{i \in I} e_i$. Then $\dim(\mathcal{H}_{\mathcal{B}}) = \deg(\mathcal{L}^{-1}) = |\mathcal{F}|$.

(For experts:
$\mathcal{H}_{\mathcal{B}}$ corresponds to the d-th graded piece of the Orlik-Solomon algebra)

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The vectors v_I represent coordinate functions on $\mathcal{H}_{\mathcal{B}}$. One can right $\mathcal{H}_{\mathcal{B}}$ as the rowspan of a $|\mathcal{F}| \times |\mathcal{B}|$ matrix with columns v_I.

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Example: $d = 2, n = 4$

For $d = 2, n = 4$, $\mathcal{L}^{-1}$ generically has degree 3 and we can take

$$\begin{pmatrix} v_{14} & v_{24} & v_{34} & v_{12} & v_{13} & v_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

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If $p = p(M)$ and $q = p(\mathcal{L})$ then the Chow form of $\mathcal{L}^{-1}$ is

$\det(\sum_I \frac{p_I}{q_I} \cdot v_I v_I^T) =$

$$\det\begin{pmatrix} \frac{p_{14}}{q_{14}} + \frac{p_{12}}{q_{12}} + \frac{p_{13}}{q_{13}} & -p_{12}/q_{12} & -p_{13}/q_{13} \\ -p_{12}/q_{12} & \frac{p_{24}}{q_{24}} + \frac{p_{12}}{q_{12}} + \frac{p_{23}}{q_{23}} & -p_{23}/q_{23} \\ -p_{13}/q_{13} & -p_{23}/q_{23} & \frac{p_{34}}{q_{34}} + \frac{p_{13}}{q_{13}} + \frac{p_{23}}{q_{23}} \end{pmatrix}.$$

Generic case: non-zero Plücker coordinates

If $\mathcal{B} = \binom{[n]}{d}$ the vectors $\{v_I : I \in \mathcal{B}\}$ can be taken to be

$$v_I = e_{I\setminus\{n\}} \quad \text{for} \quad I \ni n \qquad \text{and} \qquad \sum_{k=1}^{d}(-1)^k e_{I\setminus\{i_k\}} \quad \text{for} \quad I \not\ni n.$$

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Theorem. If $\mathcal{L} \in \mathrm{Gr}(2, n)$ has no zero Plücker coordinates, then the Chow form of $\mathcal{L}^{-1}$ is

$$\sum_{T\in\mathcal{T}_n} \prod_{\{i,j\}\in T} p_{ij}(M) \cdot \prod_{\{k,\ell\}\in T^c} p_{k\ell}(\mathcal{L}),$$

where $\mathcal{T}_n$ denotes the set of spanning trees on n vertices.

Simplicial matroids

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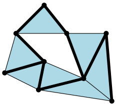
More generally, $(v_I)_{I\in\binom{[n]}{d}}$ represents the simplicial matroid of K_n^{d-1}, studied by Kalai, Bernardi–Klivans, ...



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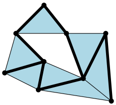

Theorem. If all Plücker coordinates of $\mathcal{L} \in \mathrm{Gr}(d,n)$ are non-zero, then the Chow form of $\mathcal{L}^{-1}$ in $\mathbb{C}[p_I(M^\perp) : I \in \binom{[n]}{d}]$ is

$$\sum_{\substack{F \text{ is a spanning} \\ \text{forest of } K_n^{d-1}}} c_F \cdot \prod_{I\in F} p_I(M) \cdot \prod_{I\notin F} p_I(\mathcal{L})$$

where $c_F \in \mathbb{Z}_+$ depends on the relative homology of F in K_{n-1}^{d-1}.

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Ex: $f = \det(\sum_i x_i v_i v_i^T)$ where $v_i \in \mathbb{R}^d$ $\longrightarrow$ subsets $I \subset [n]$ for which $\{v_i : i \in I\}$ is a basis of $\mathbb{R}^d$

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Thanks!

References

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$\mathcal{L}$ d-dim (real) linear space in $\mathbb{C}^m$

$$\mathcal{L}^{-1} = \overline{\{ (x_1^{-1}, \dots, x_m^{-1}) : x \in \mathcal{L} \cap (\mathbb{C}^*)^m \}}$$

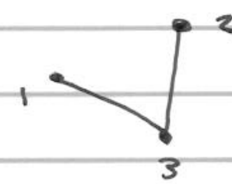
Example: $\mathcal{L} = V(x_1 + x_2 + x_3) = \text{rowspan}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

$$\mathcal{L}^{-1} = V(x_2x_3 + x_1x_3 + x_2x_3)$$

Circuits : 123

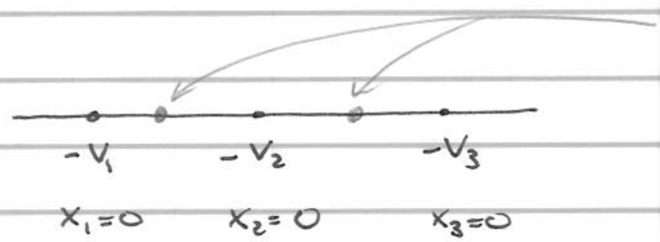
Broken circuits : 12

Broken circuit complex :



$$\mathcal{L}^{\perp} + v = \{ (t+v_1, t+v_2, t+v_3) : t \in \mathbb{C} \}$$

$\mathcal{L}^{\perp} + v$



$\mathcal{L}^{-1} \cap (\mathcal{L}^{\perp} + v)$, correspond to the roots of $\frac{d}{dt} \prod_{i=1}^{3} (t + v_i)$