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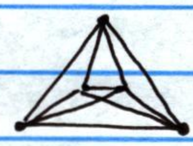
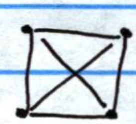
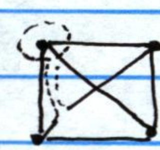
# Computing Simplicial Representatives of Homotopy classes.
(joint with M. Filakovský, P. Franek, S. Zhechev)

<u>Planarity</u> Given a graph $G$, does it embedd in the plane $G \hookrightarrow \mathbb{R}^2$ ?

 $\not\hookrightarrow \mathbb{R}^2$ ,  $\not\hookrightarrow \mathbb{R}^2$

- Hopcroft - Tarjan : Linear time planarity testing (also produces an embedding)
- Steinitz / Fáry : $G \hookrightarrow \mathbb{R}^2 \Rightarrow G \underset{LIN}{\hookrightarrow} \mathbb{R}^2$
- Hanani - Tutte : $G \hookrightarrow \mathbb{R}^2 \Leftrightarrow \exists$ map $f : G \longrightarrow \mathbb{R}^2$

(*) $|f(e_1) \cap f(e_2)| \equiv 0 \mod 2$
if $e_1 \cap e_2 = \emptyset$

 "finger move" $\rightsquigarrow$ 

does not respect this property(*)

does verify the property (*) !

Condition (*) can be tested algorithmically in linear algebra over $\mathbb{Z}_2$ ;
this gives a polynomial time planarity test

Embeddings  Given a simplicial complex $K$, $\dim K = k$, does $K \hookrightarrow \mathbb{R}^d$ ?

We can think of linear, piecewise linear or topological embeddings.

- for $d \geq 3$ "no Fáry" "$\hookrightarrow_{PL}$" $\Rightarrow$ "$\hookrightarrow_{LIN}$"

- Algorithms for testing $K \hookrightarrow_{PL} \mathbb{R}^d$ ?
  - undecidable for $d \geq 5$ and $k \in \{d-1, d\}$
  - decidable for $d = 3$
  - $\boxed{\text{polynomial-time testable } \underbrace{\text{for } d \geq \tfrac{3(k+1)}{2}}_{\text{meta stable range}}}$
  - ? in all other cases, NP-hard.

Q: (1) Can we algorithmically compute PL-embeddings?

(2) How complicated are they?

In (1), we want to compute a subdivision $K'$ of $K$ and an embedding $K' \hookrightarrow_{LIN} \mathbb{R}^d$.
For question (2) we want the number of simplices needed in $K'$.

## Deleted Products and Embeddings

Obs: $K \xhookrightarrow[f]{} \mathbb{R}^d \implies K_\Delta^{\times 2} \xrightarrow[\mathbb{Z}_2]{} \mathbb{R}^d \setminus \{0\} \simeq S^{d-1}$

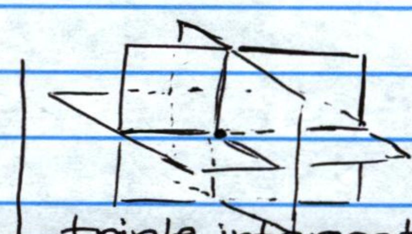
$$K_\Delta^{\times 2} = \{(x,y) \in K \times K : \begin{smallmatrix} x \in \sigma \\ y \in \tau \end{smallmatrix}, \sigma \cap \tau = \emptyset\}, \quad (x,y) \mapsto f(x) - f(y)$$

Ex:  $K$  $K_\Delta^{\times 2}$

Thm (Haefliger-Weber)
If $d \geq \frac{3(k+1)}{2}$ then $K \hookrightarrow \mathbb{R}^d \iff \underbrace{K_\Delta^{\times 2} \xrightarrow[\mathbb{Z}_2]{} S^{d-1}}$

poly-time testable
(Čadek - Krčál - Vokřínek)
actually computes
$[K_\Delta^{\times 2}, S^{d-1}]_{\mathbb{Z}_2}$.

## Higher Multiplicities


triple intersection

Thm (Mabillard - W.)
If $d \geq \frac{(r+1)k+3}{r}$, $\dim K = k$ then

$$\underset{\text{"r-embedding"}}{K \longrightarrow \mathbb{R}^d} \iff \underbrace{K_\Delta^{\times r} \xrightarrow[\mathfrak{S}_r]{} S^{d(r-1)-1}}$$

poly-time testable
Filakovský + Vokřínek

$X, Y$ simplicial complexes

$f, g : X \longrightarrow Y$ continuous maps, $f \sim g$ homotopic

$[X, Y]$ set of all homotopy classes of maps $X \to Y$

$\Big( [X,Y]_G$ " " equivariant if $G$ acts on $X$, $Y$ $\Big)$.

Ex: $[S^1, S^1] = \mathbb{Z}$ $\quad$ • $\dim X = d$, $[X, S^d] \cong H^d(X)$.
$[S^d, S^d] = \mathbb{Z}$

<u>Homotopy groups</u> $\pi_k(Y) = [S^k, Y]$

Fact: $\pi_1(Y)$ is not computable (Adian-Rabin)

Thm (Brown): If $\pi_1(Y)$ is trivial, then any higher $\pi_k(Y)$, $k \geq 2$, can be computed.

Thm (Čadek et al): • If $k$ is fixed, $\pi_k(Y)$ is poly-time computable.

• If $\dim X \leq 2d-2$, then $[X, S^d]$ is poly-time computable (for fixed $d$).

builds on work by Sergeraert

These algorithms compute $\pi_k(Y)$, $[X, S^d]$ as finitely generated abelian groups.

Example: $r=6$, $d=18$, $k=15$

$$\boxed{\begin{array}{l}\text{Özaydin '87}\\ K_\Delta^{\times r} \xrightarrow{}_{\mathfrak{S}_r} S^{d(r-1)-1}\end{array}} \xRightarrow{\text{previous thm}} K \longrightarrow \mathbb{R}^{18} \text{ r-embedding}$$

$$\underset{\text{Constraint Lemma}}{\Longrightarrow} \overset{\text{Frick}}{\sigma^{100} \longrightarrow \mathbb{R}^{19}} \text{ 6-embedding}$$

(Gromov
Blagojevic Frick Ziegler)

Q: 1) Can we compute r-embeddings?
2) How complicated are they?

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Special case of Haefliger - Weber theorem:
Van-Kampen obstruction is complete for embeddability of k-complexes into $\mathbb{R}^{2k}$, $k \geq 3$.

Thm (Freedman - Krushkal)

- If $\dim K \geq 3$, $K$ has $N$ simplices, then $K$ admits a PL-embedding into $\mathbb{R}^{2k}$ with refinement complexity $e^{N^{4+\varepsilon}}$.
- Moreover $2^{\Omega(N)}$ is sometimes necessary.

Thm: There is an algorithm that, given a finite simplicial complex $Y$ with $\pi_1(Y)$ trivial and $k \geq 2$, computes simplicial maps $g_j : \Sigma_j^k \longrightarrow Y$, $j = 1, \ldots, m$, that generate $\pi_k(Y)$.
The running time is $2^{\text{poly}(\#\text{ simplices of } Y)}$, and this is best possible.