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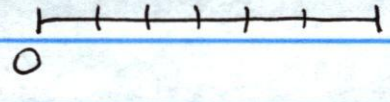
10/13/17 9:00AM

Integer points in polytopes

$P \subseteq \mathbb{R}^n$, $P = \{ \bar{x} \in \mathbb{R}^n \mid A\bar{x} \le \bar{b} \}$

Goal: $|P \cap \mathbb{Z}^n| = ??$ compute this number.

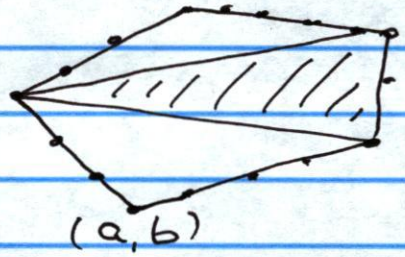
If $n=1$, 

We can compute this in poly(ℓ), ℓ = bit length of $\{A, \bar{b}\}$

$n=2$ (Pick 1899) P integer polygon.

$$|P \cap \mathbb{Z}^2| = \text{area}\, P + \frac{|\partial P \cap \mathbb{Z}^2|}{2} + 1$$

To find # of lattice pts on edge, (p,q)
compute $1 + \gcd(p-a, q-b)$.



$\Rightarrow$ there is an algorithm in polynomial time.

To calculate the area, need to triangulate P

Thm (Mordell 1951)

$$\Delta = \Delta(a,b,c) = \left\{ \begin{array}{l} x,y,z \ge 0 \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \le 1 \end{array} \right\}.$$

$$|\Delta(a,b,c) \cap \mathbb{Z}^3| = \frac{abc}{6} + \frac{ab+bc+ac+a+b+c}{4}$$
$$+ \frac{1}{12}\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b} + \frac{1}{abc}\right) - S(ab,c) - S(bc,a) - S(ac,b)$$

where $S(p,q) = \sum_{k=1}^{q} ((\frac{k}{q}))((\frac{pk}{q}))$ Dedekind sum

$$((x)) = \begin{cases} 0 & x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & \text{otherwise.} \end{cases}$$

Thm (Cayley 1857)
$A_n, B_n \subseteq \mathbb{R}^n$

$$A_n = \{ 1 \le x_1 \le 2, \ldots, 1 \le x_{k+1} \le 2x_k \quad k = 1, \ldots, n-1 \}$$

$$B_n = \{ x_1, \ldots, x_n \ge 0, \ \frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{2^2} + \ldots + \frac{x_n}{2^{n-1}} \le 2^{n-1} \}$$

Then
$$|A_n \cap \mathbb{Z}^n| = |B_n \cap \mathbb{Z}^n|$$

Note: $\operatorname{Vol} A_n = \frac{1}{n!} a_n$, a_n = # connected graphs on n vertices

$\operatorname{Vol} B_n = \frac{1}{n!} 2^{\binom{n}{2}}$ Konvolinka - P

If $P = \{ \bar{x} \in \mathbb{R}^m : A\bar{x} \leq b \}$, $\ell(P) := \sum_{ij} \lceil \log a_{ij} \rceil + \sum_i \lceil \log b_i \rceil$

$\ell = \ell(P)$ - this is the bitlength of the input.

Decision problem: $|P \cap \mathbb{Z}^m| \overset{?}{>} 0$

Counting problem: $|P \cap \mathbb{Z}^m| = ?$

in poly(ℓ).

$P := \{ x_1, \dots, x_m \geq 0 \,,\, a_1 x_1 + \dots + a_m x_m = N \}$

Knapsack polytope

Decision probl is NP-complete

Counting " #P-complete

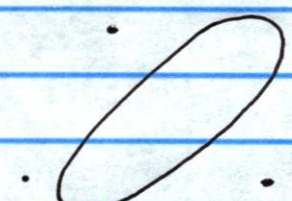
From now on, the dimension n will be bounded.

Thm (Lenstra, 1983) n fixed.

$|P \cap \mathbb{Z}^n| \overset{?}{>} 0$ is in P (can be computed in poly(ℓ) time).

Thm (Barvinok, 1993)

$|Ph \mathbb{Z}^n|$ is in P.



· flatness thm

· LLL algorithm

Operations

$P, Q \subseteq \mathbb{R}^m$ $\quad P \cup Q,\ P \cap Q,\ P \setminus Q,\ P\downarrow_{\mathbb{R}^k}$ ...

Can we compute integer points in poly time?

- $|(P \cap Q) \cap \mathbb{Z}^m|$? yes, by previous thm
- $|P \setminus Q \cap \mathbb{Z}^m|$? "
- $|(P \cap \mathbb{Z}^m)\downarrow_{\mathbb{R}^k}|$? yes, thm (Barvinok-Woods 2003)
- $|(P \cap \mathbb{Z}^m)\downarrow_{\mathbb{R}^k} \cap (Q \cap \mathbb{Z}^m)\downarrow_{\mathbb{R}^k}|$? yes,

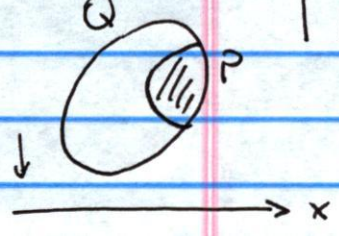
Thm (Nguyen, P.)
$P_1, P_2, \ldots, P_k \subseteq \mathbb{R}^3$

$|(P_1 \cup P_2 \cup \ldots \cup P_k) \cap \mathbb{Z}^3\downarrow_x|$ is #P-complete

⇑

Thm (Nguyen, P.)
$P \subseteq Q \subseteq \mathbb{R}^3$ convex polytopes.

$|(Q \setminus P)\downarrow_x \cap \mathbb{Z}|$ is #P-complete.



Main Results:

Thm: Given matrices A, B, C and vector b and polytopes Q, U,
Nguyen-P

$$\exists \bar{z} \in \mathbb{Z}^e \cap U \text{ s.t. } \forall \bar{y} \in Q \cap \mathbb{Z}^m \ \exists \bar{x} \in \mathbb{Z}^n :$$

$$A\bar{x} + B\bar{y} + C\bar{z} \leq b$$

Decision problem is NP-complete
Counting problem #P-complete ← (determining how many $\bar{z}$ satisfy the above conditions)

Ingredients of the proof:

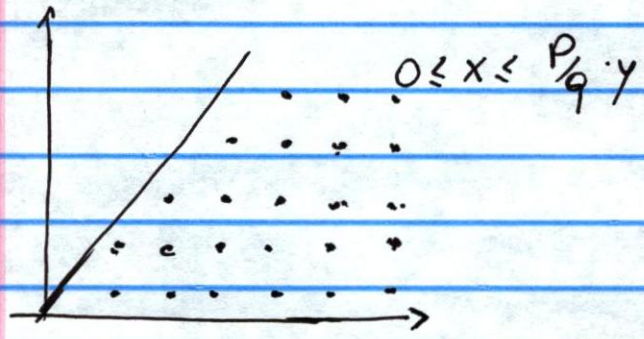
- $AP(a,b,c) = \{ a + bm, \ 0 \leq m \leq c \}$.

$$\bigcup_i AP_i(a_i, b_i, c_i) \stackrel{?}{\supseteq} \{1, \ldots, N\}$$

Thm (Stockmeyer, Meyer 1973) This problem is in coNP-complete

- continued fractions

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$



Many related problems. For example:

$P \subseteq \mathbb{R}^2$, $v \in \mathbb{R}^2$ compute

$$\min_{\lambda \in \mathbb{R}} |(P + \lambda v) \cap \mathbb{Z}^2|$$

Thm (Eisenbrand, Hähnle): This problem is NP-hard if P has many facets (edges).

Open if P has bounded number of edges.

Thm (Nguyen-P): $P \subseteq \mathbb{R}^7$ with at most 100 facets then it's NP-hard.