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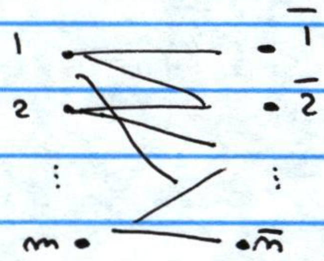
Alex Postnikov

Triangulation of root polytopes and Tutte polynomials

1. Root polytope



$G \subseteq K_{m,n}$

connected bipartite graph

$Q_G = \mathrm{conv}\,(e_i - e_{\bar{j}} \mid (i,\bar{j}) \in E(G))$

where $e_1, \dots, e_m, e_{\bar{1}}, \dots, e_{\bar{n}}$ are coordinate vectors in $\mathbb{R}^{m+n}$

$\dim Q_G = m+n-2$.

Ex: $Q_{K_{m,n}} \simeq \Delta^{m-1} \times \Delta^{n-1}$

This is related to
- tropical geometry.
- GKZ
- Tutte polynomial
- generalized permutahedra
- knot invariants

Connection to Catalan numbers: $\overline{\mathrm{Vol}}\, Q_{K_n} = C_n$
if G is not bipartite ...

$$\overline{\mathrm{Vol}}\, Q_{K_{m,n}} = \frac{(m+m-2)!}{(m-1)!\,(m-1)!} = \binom{m+n-2}{m-1}$$

Lemma: Each top dimensional simplex in a triangulation of Q_G has the form

$$\Delta(T) = \mathrm{conv}\left(e_i - e_{\bar{j}} \mid (i,\bar{j}) \in E(T)\right),$$

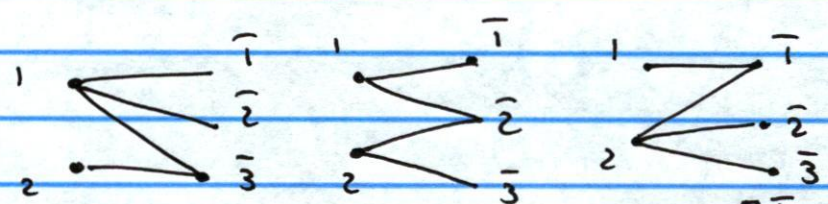
T spanning tree in G.

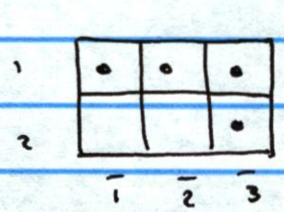
all have normalized volume 1.

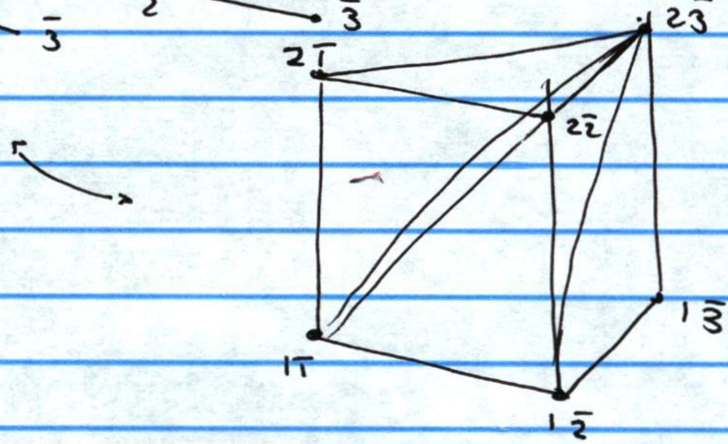
$G = K_{m,n}$ # of spanning trees $m^{m-1} \cdot n^{m-1} \geq \binom{m+n-2}{m-1}$

Take all non-crossing trees:

Ex: $m = 2$, $n = 3$





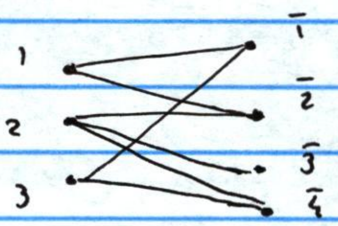


2. Generalized Permutahedra

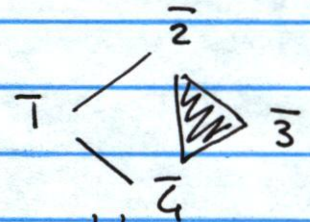
G bipartite graph, $J_i = \{ \bar{j} \in [n] \mid (i, \bar{j}) \in E(G) \}$

$\rightsquigarrow H = \{ J_1, \ldots, J_m \}$ hypergraph

Ex: $m = 3$
$n = 4$



G $\rightsquigarrow$



H

$J \subseteq [n]$, $\Delta_J = \mathrm{Conv}(e_{\bar{j}} \mid j \in J)$.

$P_G = \Delta_{J_1} + \Delta_{J_2} + \cdots + \Delta_{J_m}$

Ex: If H is a graph, P_G is a graphical zonotope
$P_G = \mathrm{Zon}_H$.

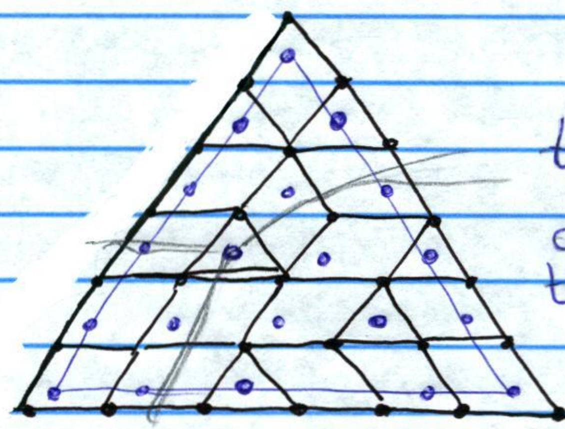
Cayley trick

$$Q_G \simeq \mathrm{conv}\left(e_1+\Delta_{J_1}, \dots, e_m+\Delta_{J_m}\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

$$Q_G \cap \left\{x_1 = x_2 = \dots = x_m = \tfrac{1}{m}\right\} = \tfrac{1}{m}\left(\Delta_{J_1}+\dots+\Delta_{J_m}\right) = \tfrac{1}{m} P_G$$

Ex: $G = K_{6,3}$ $\qquad Q_G = \Delta^5 \times \Delta^2$

$P_G = 6\Delta^2$



tropical line corresponding to each Δ

$$P_G^- = P_G - \Delta_{[m]}$$

minkowski difference: $A - B = \{c \in \mathbb{R}^n \mid B + c \subseteq A\}$

obs: $(A+B) - B = A$

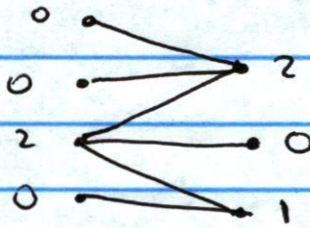
$(A-B) + B \overset{?}{=} A$ not always.

Thm: $\overline{\mathrm{Vol}}\, Q_G = \#\left(P_G^- \cap \mathbb{Z}^n\right)$

$T \subseteq G \qquad LD(T) = (a_1, \dots, a_m) \quad a_i = \deg_T(i) - 1$

$\qquad\qquad RD(T) = (b_1, \dots, b_n) \quad b_j = \deg_T(\bar{j}) - 1$.

Ex:



$LD = (0, 0, 2, 0)$

$RD = (2, 0, 1)$.

Thm: $\{T_1, \dots, T_N\}$ triangulation

$T \longrightarrow RD(T)$ is a bijection between simplices and lattice points of P_G^-.

$$\mathrm{Vol}\, Q_G = \#\left(P_G^- \cap \mathbb{Z}^n\right) \overset{\varphi_\tau}{\longleftrightarrow} \#\left(P_{G^*}^- \cap \mathbb{Z}^m\right)$$

G^* is the reflected copy of G (switching ground sets $[m]$ and $[n]$)

Thm (Galushin, Nenashev, P):
It is possible to recover the triangulation from φ_τ.

$G = K_{m,n}$

$(m-1)\Delta^{n-1} \longleftrightarrow (n-1)\Delta^{m-1}$ the matching is not trivial.

if H is a graph

$P_G = \mathrm{Zon}_H$ $\qquad$ $\#\left(P_G^- \cap \mathbb{Z}^m\right) = \#$ of trees in G.

$\updownarrow$

dual object: matroid polytope

Rest of the talk is joint with Tamás Kálmán

H hypergraph

hypertrees

$\shortparallel$

lattice points of $P_{G^*}^-$

Tutte polynomial $T_H(x,y)$;

$$T_H(x,1) = \sum_{h \in P_{G^*}^-} x^{\text{int.activ}(h)}$$

key idea of the proof:

$$T_G(x) \longleftrightarrow Q_G \longleftrightarrow T_{G^*}(x)$$

f-polynomial of a triangulation of Q_G

$f(x) \rightsquigarrow h(x)$ h-polynomial