A tale of two modules (with S. Riche)

Lecture by Geordie Williamson
Notes by Dustan Levenstein

Today: Formulas for simple and tilting characters for a reductive group $G/\mathbf{k}$ ($\mathbf{k}$ a field of positive characteristic) in terms of p-Kazhdan-Lusztig polynomials. These formulas have been proven for $p \geq h, p \geq 2h-2$, but are expected to hold in general.

These formulas imply many different numbers in modular representation theory, e.g., decomposition numbers for symmetric groups.

Example Billiards Conjecture by Lusztig-Williamson (see youtube!).

Let $\Phi^+ \subset \Phi$ be the positive roots, $\mathscr{X}_+ \subset \mathscr{X}$ the dominant weights, W_f the finite Weyl group.

Let $W = \mathbb{Z}\Phi \rtimes W_f$ be the (dual) affine Weyl group, containing the set $\mathcal{S}$ of simple reflections.

Let H be the affine Hecke algebra,

$$H = \bigoplus_{x \in \mathbb{N}} \mathbb{Z}[v^{\pm 1}]\delta_x$$

with $(\delta_s + v)(\delta_s - v) = 0$, and $H_f \subset H$.

Define H_f-modules $\mathrm{triv}_v : \delta_s \mapsto v^{-1}$ and $\mathrm{sgn}_v : \delta_s \mapsto -v$ for $s \in \mathcal{S}_f$.

Let

$$M^{\mathrm{sph}} = \mathrm{triv}_v \otimes_{H_f} H = \bigoplus_{x \in {}^f W} \mathbb{Z}[v^{\pm 1}]\mu_x$$

be the spherical module, and

$$M^{\mathrm{asph}} = \mathrm{sgn}_v \otimes_{H_f} H = \bigoplus_{x \in {}^f W} \mathbb{Z}[v^{\pm 1}]\eta_x$$

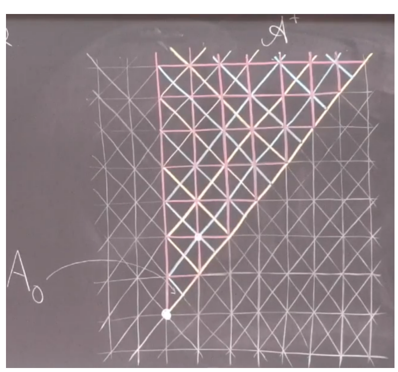
be the antispherical module.

Let $\mathcal{A}$ be the alcoves in $\mathscr{X}_{\mathbb{R}} := \mathscr{X} \otimes_{\mathbb{Z}} \mathbb{R}$, with $A_0 \in \mathcal{A}$ fundamental alcove.

We have

$$W \simeq \mathcal{A}$$

given by $w \mapsto w(A_0)$, and ${}^f W \simeq \mathcal{A}^+$ dominant alcoves.



Set $b_s = \delta_s + v$ (KL generator) for $s \in \mathcal{S}_f$.

$$\mu_x \cdot b_s = \begin{cases} \mu_{xs} + v\mu_x & \text{if } xs > x, xs \in {}^fW, \\ \mu_{xs} + v^{-1}\mu_x & \text{if } xs < x, xs \in {}^fW, \\ (v+v^{-1})\mu_x & \text{if } xs \notin {}^fW, \end{cases}$$

and

$$\eta_x b_s = \begin{cases} \mu_{xs} + v\mu_x & \text{if } xs > x, xs \in {}^fW, \\ \mu_{xs} + v^{-1}\mu_x & \text{if } xs < x, xs \in {}^fW, \\ 0 & \text{if } xs \notin {}^fW. \end{cases}$$

Next we define the **Hecke Category**:

Let $\mathcal{H}$ be the graded (with shift $B \mapsto B(1)$) monoidal category generated by B_s, $s \in \mathcal{S}$, presented by generators and relations. It is isomorphic to Bott-Samelson $\mathbb{Z}$-sheaves on the affine flag variety of the dual group. We always have $H = [\mathcal{H}]_\oplus$ split Grothendieck group (no Krull-Schmidt theorem, hom spaces are free of finite rank over $\mathbb{Z}$).

But for any field $\mathbf{k}$, we can consider $\mathbf{k} \otimes \mathcal{H}$, which has Krull-Schmidt, so the indecomposable objects $\{{}^pB_x \mid x \in W\}$ give a basis of H.

So we get the p-canonical basis $\{{}^pb_x\} \subset H$.

Facts:

(0) $p = \operatorname{char} \mathbf{k}$, basis only depends on p, not on $\mathbf{k}$.

(1) (hard) 0-canonical basis = Kazhdan-Lusztig basis.

(2) ${}^pb_x = b_x + \sum_{y<x} {}^pa_{y,x} b_y$ with ${}^pa_{y,x} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

(3) Very rich combinatorics!

(4) Completely known for $A_{\leq 8}, D_{\leq 6}, \widetilde{A}_1$, ...

Later we will use

$${}^pb_x = \sum h_{y,x}\delta_x,$$

where we call the $h_{y,x}$ the p-Kazhdan-Lusztig polynomials.

Let w_f be the longest element of W_f. We define categorifications:

$$\mathcal{M}^{\text{sph}} := B_{w_f} \cdot \mathcal{H}$$

categorifies $M^{\text{sph}} = b_{w_f} \cdot H$ and

$$\mathcal{M}^{\text{asph}} := \mathcal{H}/\langle B_s B_t \cdots B_u \mid s \in \mathcal{S}_f\rangle_\oplus$$

categorifies $M^{\text{sph}} = H/\langle b_s H \mid s \in \mathcal{S}_f\rangle$.

We have $[\mathcal{M}^{\text{sph}}]_\oplus = M^{\text{sph}}$ and $[\mathcal{M}^{\text{asph}}]_\oplus = M^{\text{asph}}$, from which we obtain p-canonical bases, ${}^pb_x^{\text{asph}}$, ${}^pb_x^{\text{sph}}$, and p-Kazhdan-Lusztig polynomials ${}^pm_{y,x}$ (spherical) and ${}^pn_{y,x}$ (anti-spherical).

Let $G/\mathbf{k}$ be split semisimple, simply connected with root system Φ. For $\lambda \in \mathscr{X}$ we associate L_λ simple, Δ_λ standard, T_λ tilting modules. We assume $\operatorname{char} \mathbf{k} \geq h$.

We have $W \simeq p\mathbb{Z}\Phi \rtimes W$ acting on $\mathscr{X}$ via the p-dot action $(x, \lambda) \mapsto x \bullet_p \lambda$.

$$\text{Rep}_0 := \langle L_{x\bullet 0} \mid x \in^f W\rangle$$

principal block, containing Tilt_0. There is a well crossing functor Θ_s acting on this as well.

We relabel $L_x := L_{x\bullet 0}$, $T_x := T_{x\bullet 0}$, etc., also L_A and T_A for $A \in \mathcal{A}^+$.

We have

$$[\Delta_x \cdot \Theta_s] = \begin{cases} [\Delta_x] + [\Delta_{x\bullet s}] & \text{if } xs \in^f W \\ 0 & \textit{else}. \end{cases}$$

Hence

$$[\mathrm{Rep}_0] \cong M^{\mathrm{asph}}_{v=1}.$$

Theorem 0.1 *(Riche-Williamson, Elias-Losev, Achar-Makisumik-Riche-Williamson)*

$$\mathbf{k} \otimes \mathcal{M}^{\mathrm{asph}} \simeq \mathrm{Tilt}_0$$

$$B_x^{\mathrm{asph}} \mapsto T_x.$$

when we forget the grading.

Remark The proof in general uses Achar-Riche and Koszul duality, $\mathcal{M}^{\mathrm{asph}} \leftrightarrow \mathcal{M}^{\mathrm{sph}}$.

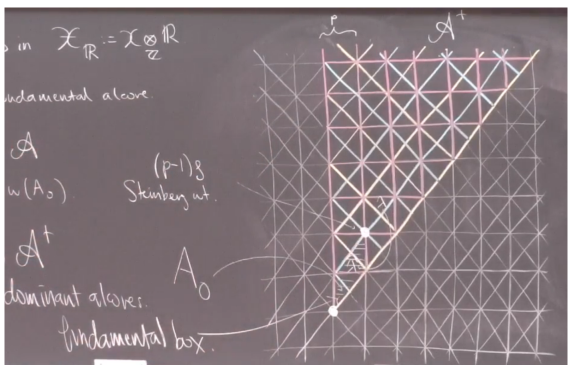
$p \geq h$:

Corollary 0.2

$$(T_x : \Delta_y) =^p n_{y,x}(1)$$

True for all p in type A, Elias-Losev. We conjecture true in general.

Consider GT-modules ($GT = (\mathfrak{g}, T)$): $\widehat{L}_\lambda, \widehat{\Delta}_\lambda, \widehat{Q}_\lambda$ indecomposable/projective hull/cover of $\widehat{L}_\lambda$, for $\lambda \in \mathscr{X}$.



Type D: $\widehat{Q}_{\hat{\lambda}} = T_\lambda|_{G,T}$ (known for $p \geq 2h-2$). This gives a formula for simple characters in principle.

Problem: not practical.

$$c = \sum_{\alpha \in \Phi} \mathrm{ht}(\alpha) = \langle \rho, \rho^\vee \rangle$$

We need to calculate p-canonical elements ℓ hom C: $2C - \ell(w_f)$. We wish to explain how to use the spherical module to reduce to ℓ: $0 - C - \ell(w_f)$.

Consider:

$$T_{A_0+\rho} = \mathrm{Trans}^{p\rho}_{(p-1)\rho}(\text{Steinberg module}).$$

Hence we get a map

$$M^{\mathrm{sph}}_{v=1} \hookrightarrow [\mathrm{Tilt}_0] = [\mathrm{Rep}_0]$$

$$\mu_{id} \mapsto [T_{A_0+\rho}]$$

This categorifies

$$\mathbf{k} \otimes \mathcal{M}^{\mathrm{sph}} \to \mathbf{k} \otimes M^{\mathrm{asph}} \cong \mathrm{Tilt}_0\,.$$

T_A is the image of $\varphi \iff A \in \mathcal{A}^{++}$

$$J : \left[\substack{\mathbf{k} \otimes \mathcal{M}^{\mathrm{sph}} \\ \text{with grading forgotten}}\right] \hookrightarrow \mathbb{Z}W$$

(full but not faithful)

Lemma 0.3 *Suppose $\varphi(m) = [T_A]$, $p \geq 2h - 2$, $x(A_0)$ is in the fundamental box:*

$$\left(\widehat{Q}_{(\rho+A_0)w_0x} : \widehat{\Delta}_{(\rho+A_0)y}\right) =^p h_{y,w_0x}(1).$$

Remark (1) Because w_0x is maximal in its left W_f-coset, all polynomials on RHS are seen in the spherical module.

(2) Suitable singular versions have a chance to hold for all p.

(3) Generalizes and represents Fiebig (MSRI, 2008).

$$^p m_{y,x} = m_{y,x}$$

for all x, y in the fundamental box implies Lusztig's character formula.

Slogan: Original formulation of Lusztig character formula does not hold with KL replaced by p-KL. G, T version does!

Remark Should be a tale of three modules; periodic module is missing.