On Picard groups of blocks of finite group algebras

Lecture by Radha Kessar
Notes by Dustan Levenstein

Joint with Robert Boltje and Markus Linckelmann.

Let B be an algebra. We say M a (B, B)-bimodule is **invertible** if M is a finitely generated projective module as a left/right B-module and there is a (B, B)-bimodule, f.g., left/right projective, such that

$$M \otimes_B N \cong B \cong N \otimes_B M$$

as (B, B)-bimodules.

We have the Picard group

$$\mathrm{Pic}(B) = \{\text{iso-classes } [M] \text{ of invertible } (B, B)\text{-bimodules}\},$$

$$[M] \cdot [N] = [M \otimes_b N],$$

with unit given by $[B]$.

We have

$$\mathrm{Out}(B) \hookrightarrow \mathrm{Pic}(B)$$

$$[\alpha] \mapsto [B_\alpha],$$

where $B_\alpha = B$ as left module and multiplication on the right $b \cdot x = b\alpha(x)$ twisted by α for $b, x \in B$.

Let p be a prime number, $\mathcal{O} \geq \mathbb{Z}_p$ discrete valuation ring. We say $\mathcal{O}$ is <u>p-adic</u> if $\mathcal{O}$ is finitely generated as a $\mathbb{Z}_p$-module, equivalently $k = \mathcal{O}/J(\mathcal{O})$ is finite (of char p). (Here $J(\mathcal{O})$ denotes the maximal ideal and $\mathcal{O}/J(\mathcal{O})$ is the residue field.)

Theorem 0.1 *(Curtis-Reiner) If $\mathcal{O}$ is p-adic and G is a finite group, then $\mathrm{Pic}(\mathcal{O}G)$ is a finite group.*

There are other versions, e.g., $\mathcal{O}$ can be a Dedekind domain.

Theorem 0.2 *Suppose that $k = \overline{\mathbb{F}_p} = \mathcal{O}/J(\mathcal{O})$. Then $\mathrm{Pic}(\mathcal{O}G)$ is the colimit of $\mathrm{Pic}(\mathcal{O}_o G)$, where $\mathcal{O}_o$ runs over all p-adic subrings of $\mathcal{O}$.*

Brauer: If $\mathrm{char}(K) = 0$ and K contains all p' roots of unity, then the Schur index of any absolutely irreducible character of a finite group over K is trivial.

Theorem 0.3 *(Florian Eisele, 2018)*
Suppose $k = \overline{\mathbb{F}_p}$ and $\mathcal{O}$ is unramified. Then $\mathrm{Pic}(\mathcal{O}G)$ has the structure of an algebraic group over k.

Theorem 0.4 *(Roggenkamp-Scott, 1987) Suppose P is a finite p-group. Then $\mathrm{Pic}(\mathcal{O}P) \cong \mathrm{Hom}(P, \mathcal{O}^\times) \rtimes \mathrm{Out}(P)$.*

Example $p = 2$, $P = C_2 \times C_2$, $\mathrm{Pic}(\mathcal{O}P) \cong (C_2 \times C_2) \rtimes S_3$.
Contrast: $k = \mathcal{O}/J(\mathcal{O})$ $\mathrm{Pic}(kP) \supset GL(2, k)$.

Example $G = M_{11}$, $P = 3$, $\mathrm{Pic}(B) = \{1\}$ is trivial for B the principal block of $\mathcal{O}G$ (this example is due to Charles Eaton).

Now let H be a finite group, X an indecomposable $\mathcal{O}H$-module, finitely generated $\mathcal{O}$-free.

Definition A **vertex** of X is a subgroup $R \leq H$ minimal wrt the property that X is a direct summand of

$$\operatorname{Ind}_R^H \operatorname{Res}_R^H X,$$

and R is a p-subgroup.

A **source** is an indecomposable $\mathcal{O}R$-module such that X is a summand of $\operatorname{Ind}_R^H V$, R vertex. We refer to (R, V) as a **vertex source pair**.

We say that X is **trivial source** (ts) if $V = \mathcal{O}$, **linear source** (ls) if $\operatorname{rank}_{\mathcal{O}} V = 1$, and **endopermutation source** (eps) if V is an endopermutation $\mathcal{O}R$-module, that is, $V \otimes_{\mathcal{O}} V^*$ is a permutation $\mathcal{O}R$-module.

Let B be a block of $\mathcal{O}G$, G a finite group, then B is indecomposable as an $\mathcal{O}[G \times G]$-module, has a vertex and a source.

We have

$$\tau(B) \subseteq \mathcal{L}(B) \subseteq \mathcal{E}(B) \subseteq \operatorname{Pic}(B),$$

where

$$\tau(B) = \{[M] \in \operatorname{Pic}(B) : M \text{ is ts as an } \mathcal{O}[G \times G]\text{-module}\},$$
$$\mathcal{L}(B) = \{[M] \in \operatorname{Pic}(B) : M \text{ is ls as an } \mathcal{O}[G \times G]\text{-module}\},$$
$$\mathcal{E}(B) = \{[M] \in \operatorname{Pic}(B) : M \text{ is eps as an } \mathcal{O}[G \times G]\text{-module}\}.$$

Example $\operatorname{Pic}(\mathcal{O}P) = \mathcal{L}(\mathcal{O}P)$ in the conditions of Theorem 0.4.

Let B be a block of $\mathcal{O}G$. We associate to B the **defect group** $P \leq G$, a p-subgroup, and **Inertial quotient** E, a p'-group, $E \subseteq \operatorname{Out}(P)$.

Example If B is the principal block, then P is a Sylow-p-subgroup and $E \cong N_G(P)/PC_G(P)$.

Theorem 0.5 *(BLK '18)*

Assume $\mathcal{O}/J(\mathcal{O}) = \overline{\mathbb{F}_p}$. Suppose that B has abelian defect group P and abelian inertial quotient E acting fixed-point-freely on $P \setminus \{1\}$.

Then $\mathcal{E}(B) = \operatorname{Pic}(B) \hookrightarrow \operatorname{Hom}(P \rtimes E, \mathcal{O}^\times) \rtimes N_{\operatorname{Aut}(P)}(E)$.

Ingredients of proof:

- Stable equivalence (Puig 1999),
- Structure of Stable Picard group (Carlson-Rouquier 2000),
- Structure of $\operatorname{Pic}(\mathcal{O}P \rtimes E)$ (Hertweck-Kimmerle '02, Zhou 2005). This step utilizes the Weiss criterion.

Let B be a block of $\mathcal{O}G$, P defect group, E inertial quotient. Let $\mathcal{F} : \mathcal{F}_P(B)$ be the fusion system (category with objects subgroups of P, morphisms given by certain injective group homomorphisms).

Let

$$\operatorname{Aut}(P, \mathcal{F}) \leq \operatorname{Aut}(P)$$

be "those automorphisms which stabilize $\mathcal{F}$",

$$\operatorname{Out}(P, \mathcal{F}) = \operatorname{Aut}(P, \mathcal{F}) / \operatorname{Inn}(P),$$

$\mathcal{D}_{\mathcal{O}}(P)$ the Dade group of endopermutation $\mathcal{O}P$-modules, $\mathcal{D}_{\mathcal{O}}(P, \mathcal{F}) \leq \mathcal{D}_{\mathcal{O}}(P)$ the $\mathcal{F}$-stable Dade group of P. We have an action of $\operatorname{Out}(P, \mathcal{F})$ on $\mathcal{D}_{\mathcal{O}}(P, \mathcal{F})$.

Theorem 0.6 *(BLK) $\mathcal{O}/J(\mathcal{O}) = \overline{\mathbb{F}_p}$. There is a group homomorphism*

$$\Phi : \mathcal{E}(B) \to \mathcal{D}_{\mathcal{O}}(P, \mathcal{F}) \rtimes \operatorname{Out}(P, \mathcal{F})$$

with kernel isomorphic to a subgroup of $\operatorname{Hom}(E, \mathcal{O}^\times)$. Also $\mathcal{E}(B)$ is finite.

- $$\Phi(\mathcal{L}(B)) \subseteq Hom(P/Foc(\mathcal{F}), \mathcal{O}^\times) \rtimes \mathrm{Out}(P, \mathcal{F}).$$

- $$\Phi(T(B)) \subseteq \mathrm{Out}(P, \mathcal{F}).$$

The proof of this theorem utilized results of Scott and Puig.