Overgroups of regular unipotent elements, finite and algebraic

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Let G be a simple linear algebraic group over $k = \overline{k}$ algebraically closed, $\mathrm{char}(k) = p \geq 0$, adjoint.

Recall an element $u \in G$ is unipotent if $u - 1$ is nilpotent when you embed G in GL. The element $x \in G$ is regular if $\dim C_G(x) = \mathrm{rk}\, G$.

Theorem 0.1 *(Steinberg) Regular unipotent elements exist and they are dense in the variety of unipotent elements. The centralizer $C_G(x)$ is unipotent and abelian.*

Question: For $G = G_2, F_4, E_6, E_7, E_8$ (exceptional types), $X = PSL_2(q)$, $q = p^a$. Suppose $X < G$ containing a regular unipotent element of G. Question: Does there exist $A < G$, A simple of type A_1 with $X < A$?

Answers: 1990: (G classical) (any unipotent) Sertz-Testerman if X is not contained in any proper parabolic subgroup of G and if $p > 19, 43, 43, 67, 113$ respectively: yes.

1997: semiregular: If $q > p$ then yes. If $q = p$ and $N_G(X) \geq PGL_2(p)$ then yes. If $G = G_2$, yes.

Guralnick-Malle: List of possible maximal Lie primitive subgroups of G (exceptional) containing a regular unipotent element, $PSL_2(p)$. The order of a regular element being p implies $p \geq 13, 13, 19, 31$. What about $13 \leq p \leq 43$, $19 \leq p \leq 43$, $19 \leq p \leq 67$, $31 \leq p \leq 113$ respectively?

More general question: Let $Y = H(p^a) < G(p^b)$ be an embedding of finite groups of Lie type in characteristic $p > 0$. Let $\overline{H}$ and $\overline{G}$ be algebraic groups over $\overline{k}$, such that $Y = \overline{H}^{\sigma}$, $G(p^b) = \overline{G}^{\delta}$ for some endomorphisms σ, δ. When does there exist an appropriate embedding of $\overline{H} < \overline{G}$ with $\overline{H}$ δ-invariant?

Liebeck-Saxl-Testerman showed: If $\mathrm{rank}(\overline{H}) > \frac{1}{2}\mathrm{rank}(\overline{G})$ and $p^a > 2$, then yes, with 4 exceptions from $q = 3$ or 5.

Liebeck-Seitz 1998: If $\overline{G}$ is exceptional with $q > 9$, and $Y \neq A_1(q), B_2(q), G_2(q), A_2^{\epsilon}(16)$, then there is a $\overline{Y} < \overline{G}$, $Y < \overline{Y}$ such that every Y-invariant subspace of $\mathrm{Lie}(G)$ is $\overline{Y}$-invariant. For the remaining groups if $q > 24, 134, 248, 776, 2624$ respectively, then they get the same statement.

Theorem 0.2 *(Burness-Testerman) Let G be a simple exceptional algebraic group of adjoint type over k, $\mathrm{char}(k) = p > 0$. Let $X = PSL_2(p)$, $X < G$, containing a regular unipotent.*

Then exactly one of the following:

(1) $X < A$ with $A < G$ closed simple A_1-subgroup of G,

(2) $G = E_6$, $p = 13$, $X < D_5$ parabolic subgroup of G, or

(3) $G = E_7$, $p = 19$, $X < E_6$ parabolic subgroup of G.

Proposition 0.3 *Let G be as above. Let $X = PSL_2(p)$, $X < G$ containing regular unipotent element and $A < G$ closed simple A_1-subgroup with $X < A$. Then X does not lie in a proper parabolic subgroup of G.*

Craven: Let F be a finite almost simple group defined over a field of characteristic p with socle $F_4(p^b)$, $E_6(p^b)$, $E_6(p^b)$, $E_7(p^b)$. Let $M < F$ be maximal such that $M = N_F(X)$ with $X = PSL_2(p^a)$.

For F_4, M is determined unless $p^a = 9$, or $p^a = 13$ and X contains a regular unipotent.

For E_6, there is no such M.

For E_7, M is determined unless $p^a = 7, 8, 25$, or $p^a = 19$ and X contains a regular unipotent.

Craven's counterexamples to extension: E_6 for $p = 13$, $Y = PSL_2(13)$, $k = \overline{k}$. Y has a 9-dimensional irreducible orthogonal representation over k, $Y < B_4 < D_5$.

The spin representation for D_5 restricted to Y, $11 \oplus 5$ where 11 is the simple 11-dimensional irreducible kY-module, $\dim H^1(Y, S) = 1$. Look at YS, there is an $X < YS$ complement to S not conjugate to Y.

In E_6, $P = D_5$ parabolic, $P = LQ$, $L' = D_5$, $Q =$ spin module for L'. We have $X = PSL_2(13) < LQ$. Then show X contains a regular element.

Contrast with algebraic groups. Take G simple adjoint algebraic group over k. In 1997, Saxl-Seitz classified maximal positive-dimensional subgroups of G containing a regular unipotent.

Either M is a maximal parabolic or M is on a (relatively) short list of reductive groups. For example, for G exceptional,

- $M = A_1 < G$, $p \geq h$ or $p = 0$,
- $M = A_2 \cdot 2 < G_2$, $p = 2$
- $M = F_4 < E_6$,
- $D_4 \cdot S_3 < F_4$, $p = 3$
- $(D_4 T_2) S_3 < E_6$, $p = 3$,
- $(E_6 T_1) \cdot 2 < E_2$, $p = 2$,
- $(A_1)^7 PGL_3(2) < E_7$, $p = 7$.

Theorem 0.4 *(Zalesski-Testerman) Let G be as above and let $H < G$ be a closed connected reductive (positive dimensional) subgroup of G containing a regular unipotent element. Then H does not lie in a proper parabolic subgroup of G.*

As a corollary, we get a list of connected reductive overgroups of regular unipotents.