Fock space categorification, Soergel bimodules, and modular representation theory in type A

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We care about decomposition numbers in Modular Representation Theory (MRT). Many interesting categories in MRT can be packaged into a categorical representation of $\widehat{\mathfrak{sl}}_e$ (e.g., $e = p$, or $q^e = 1$) categorifying a Fock space.

For the novice:

Ordinary representation of $\mathfrak{sl}_2$: We have weight spaces $V[n]$ with raising and lowering operators.

$$V[-2] \underset{f}{\overset{e}{\rightleftarrows}} V[0] \underset{f}{\overset{e}{\rightleftarrows}} V[2]$$

A categorical representation consists of categories and functors,

$$\mathcal{V}[-2] \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{V}[0] \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{V}[2]$$

Naively we would simply enforce that they satisfy the Lie algebraic relations.

There are two families of morphisms that Chuang-Rouquier '06 specify, one from specific natural transformation $NH_K \to \mathrm{End}(E^k)$, and the other from bi-adjunctions between E and F.

You might be familiar with the famous proof:

$$D^b(\mathcal{V}[-k]) \cong D^b(\mathcal{V}[k])$$

They also proved **rigidity**: (In reps of $\mathfrak{sl}_2$, $V_\lambda^{\oplus m} \cong V_\lambda \boxtimes V[\lambda]$)

If $\mathcal{V}, \mathcal{W}$ are cat $\mathfrak{sl}_2$ reps, with

$$[\mathcal{V}] \cong [\mathcal{W}] \cong V_k^{\oplus m}$$

isotypic, and

$$\mathcal{V}[k] \cong \mathcal{W}[k]$$

then $\mathcal{V} \cong \mathcal{W}$.

This suggests an approach to MRT:

- Prove rigidity for categorifications of Fock space,
- Find another Fock space categorification where we can compute!

There are several problems with this outline of an approach.

Problem 1: There is a category where we can compute: The Hecke category, aka singular Soergel bimodules. The Grothendieck group is NOT a Fock space. So let's explore how they might be connected, to get a sense for what we need to do to relate them.

1 Weird Representation Theory

Pick $e \geq 2$. Let $\mathcal{P}$ be the set of partitions $\{\lambda_1 \geq \lambda_2 \geq \ldots \geq 0\}$ (non-increasing sequences of non-negative integers which eventually become zero).

Let $F = \operatorname{Span} \mathcal{P}$.

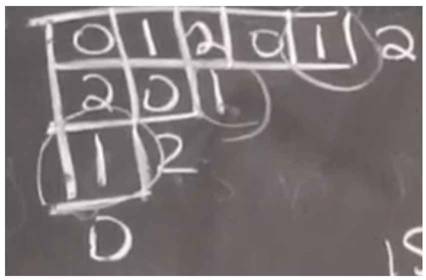
For $i \in \mathbb{Z}/e\mathbb{Z}$, and define

$$f_i(\lambda) = \sum_{\mu = \lambda + \boxed{\text{i}}} \mu,$$

$$e_i(\lambda) = \sum_{\mu = \lambda - \boxed{\text{i}}} \mu,$$

where boxes are labeled by their contents modulo e.

Example For $i = 1$, $e = 3$,

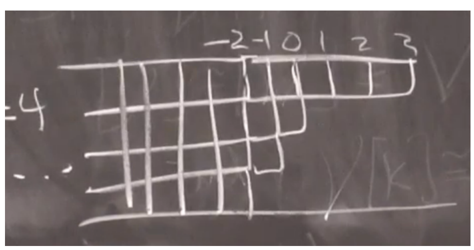


$$\operatorname{Span}\{\lambda \pm \boxed{\text{i}} \pm \boxed{\text{i}} \pm \boxed{\text{i}}\}$$

is an 8-dimensional subrepresentation of F for $\langle e_1, f_1 \rangle \cong \mathfrak{sl}_2$ isomorphic to $V_1^{\otimes 3}$, where V_1 is the standard 2-dimensional representation.

Also fix $m \geq 1$. Let $\mathcal{VP}_m$ be the set of virtual partitions with m rows, $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\}$ (not necessarily ≥ 0).

Example For $m = 4$,



We define $V\mathcal{P}_m = \operatorname{Span} \mathcal{VP}_m$.

For example, $V\mathcal{P}_1 \cong \mathbb{C}^{\mathbb{Z}} = \mathbb{C}^e[t, t^{-1}]$ standard "level 0" representation of $\widehat{\mathfrak{sl}}_e \twoheadrightarrow \mathfrak{sl}_e[t, t^{-1}]$.

$$V\mathcal{P}_m = \Lambda^m(V\mathcal{P}_1).$$

$F = \Lambda^{\frac{\infty}{2}}(V\mathcal{P}_1)$ is the "level 1" representation, and the $V\mathcal{P}_m$ are level 0. Level 1 has a highest weight vector, but level 0 does not.

Let $F_{\leq n}$ be the subspace spanned by $\mathcal{P}_{\leq n}$. There are vector space maps

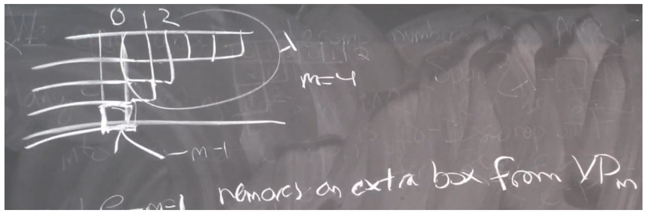
$$F_{\leq m-1} \hookrightarrow F_{\leq m} \overset{\psi}{\hookrightarrow} V\mathcal{P}_m$$

and we care about the composition φ.

Here $\mathrm{Im}(\varphi)$ is the subspace of $V\mathcal{P}_m$ where $\lambda_m = 0$.

We have φ does intertwine in a restricted sense $\psi f_i = f_i \varphi$, with $f_i(F_{\leq m-1}) \subset F_{\leq m}$.

Both subspaces $F_{\leq m-1}$, $\mathrm{Im}(\varphi)$ are preserved by e_i for $i \neq -m-1$.



e_{-m-1} removes an extra box from $V\mathcal{P}_m$.

So φ is a restricted, **truncated** intertwiner, by which we mean we truncate the action of e_{-m-1} to ignore the "frozen box".

Accept for the moment that we can do this. Moral: A piece of F and a piece of $V\mathcal{P}_m$ are isomorphic if you're a little weird.

Recall that, for $\mathfrak{sl}_2 = \langle e_{-m-1}, f_{-m-1} \rangle$ (henceforth $e = e_{-m-1}, f = f_{-m-1}$), $F/V\mathcal{P}_m$ splits into

$$\bigoplus V_1^{\otimes k}.$$

$$V_1 = v_{\square} \underset{f}{\overset{e}{\rightleftarrows}} v_{\emptyset}$$

Think of the frozen box as the last tensor factor.

Let M be any $\mathfrak{sl}_2$ representation. Consider $M \otimes V_1$.

$$\varphi : M \to M \otimes V_1,$$
$$m \mapsto m \otimes v_{\square}.$$

$$f(m \otimes v_{\square}) = f(m) \otimes v_{\square} + \cancel{m \otimes f(v_{\square})}$$
$$e(m \otimes v_{\square}) = e(m) \otimes v_{\square} + m \otimes v_{\emptyset}$$

Let $\underline{e}$ on $M \otimes v_{\square}$ be $\underline{e}(m \otimes v_{\square}) = e(m) \otimes v_{\square}$; $\underline{e}$ makes ϕ an intertwiner.

To describe $\underline{e}$ without reference to φ, M on M, there exists a pairing $(,)$ for which e, f are biadjoint. So $\underline{e}$ is adjoint to f on $M \otimes v_{\square}$ with respect to $(,)$ induced from M.

Let's return to the approach to studying MRT outlined above.

1.1 Filtrations

Chuang-Rouquier proved **filteredness**: If

$$[\mathcal{V}] = V = \bigoplus V_n^{\oplus \mu_n},$$

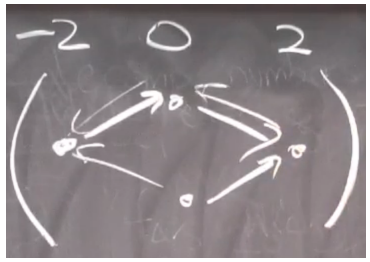
then $\mathcal{V}$ has a filtration by Serre subcategories $\mathcal{V}_{\geq n}$ such that

$$[\mathcal{V}_{\geq n+1}/\mathcal{V}_{\geq n}] \cong V_n^{\oplus \mu_n}.$$

Example

$$V = V_1^{\otimes 2}$$

is given by



which is isomorphic through a change of basis to the direct sum of a 1-dimensional and a 3-dimensional irreducible,

But you can't do this change of basis on a categorical level. Instead there is a filtration in which the 3-dimensional irreducible occurs as a Serre subcategory.

In particular the weight space $\mathcal{V}[0]$ is filtered in an interesting way.

The **Big idea**: Rigidity fails for non-isotypic, so to try to rigidify, we need to understand the filtration. In the examples, the filtration $\mathcal{V}_{\geq n}[k]$ is a highest weight categorical filtration!!

Example Categorification of Schur-Weyl duality

$$\mathcal{V}[0] \cong \mathcal{O}_0.$$

Ben Webster built a categorification of tensor products of irreducible representations whose highest weight structure were naturally present.

Losev, Losev-Webster proved rigidity of categorifications of $V_1^{\otimes n}$ in the presence of compatible highest weight structures.

Part I of paper: Construct a categorical representation of $\widehat{\mathfrak{sl}}_e$ on "SSBim" for $\widehat{\mathfrak{gl}}_m$, a "categorified affine Schur-Weyl duality".

Showed that this is a "cellular category" from which we can obtain highest weight structures.

Massage a lot (construct a parabolic version, take Ringel duality, ...) to get a categorification of $V\mathcal{P}_m$. The massaging is made technically possible because of the highest weight structures.

Part II of paper: $V\mathcal{P}_m$ vs Fock space.

Losev's categorical truncation: Given a categorification $\mathcal{V}$ of $M \otimes V_1$, with compatible highest weight structures, then $M \otimes v_{\square}$ is categorified by a highest weight subcategory $\mathcal{W}$.

Its hom form descends to $(,)$ on $M \otimes v_{\square}$.

So F on $\mathcal{V}$ restricts to the desired F on $\mathcal{W}$. Let $\underline{E}$ be the adjoint.

Theorem 1.1 *This works:* $(F, \underline{E})$ *gives a categorical* $\mathfrak{sl}_2$ *representation on* $\mathcal{W}$.

Moral: Highest weight structures do the work!

Eventually, with some very technical details, we prove:

- Rigidity for categorifications of pieces of F with compatible highest weight structures.
- Highest weight subquotients of "$\mathrm{SSBim}(\mathfrak{gl}_m)$" and MRT Fock categorifications are equivalent.

1.2 Quick Description of Singular Soergel Bimodules

There is an action on $\mathbb{Z}^m$ by the affine Weyl group $W_{\text{aff}} \supset S_m$, so $s_0(n_1, \dots, n_m) = (n_m - e, \dots, n_1 + e)$.

$$\Lambda = \{n_1 \le n_2 \le \cdots \le n_m \le n_1 + e\}$$

is the fundamental domain.

$\mathrm{Stab}(\underline{n})$ is a proper parabolic subgroup of W_{aff}.

Let $R = \mathbb{C}[x_1, \dots, x_m, y]$ acted on by W_{aff}. Then

$$R^{\mathrm{Stab}(\underline{n})} \subset R$$

is Frobenius.

For a Frobenius extension, induction and restriction are nice functors. Singular Soergel bimodules are obtained by induction-restriction-induction-restriction... between the stabilizers.