Lie superalgebras and 2-representation theory

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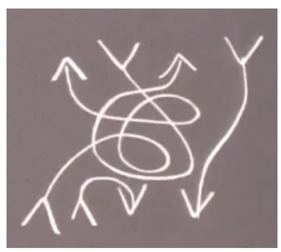
We will work over a field $\mathbf{k} = \overline{\mathbf{k}}$, $\operatorname{char} \mathbf{k} \neq 2$.
We introduce the **oriented Brauer category**, for $\delta \in \mathbf{k}$, denoted

$$\mathrm{OB}(\delta).$$

It is a tensor category, with objects given by words in $\wedge, \vee$, e.g.,

$$\wedge \wedge \vee = \wedge \otimes \wedge \otimes \vee,$$

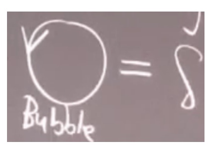
and morphisms given by oriented Brauer diagrams up to isotopy (from bottom to top), for example, the following is a morphism from $\wedge \wedge \vee\vee$ to $\wedge \vee \wedge\vee$:



Composition of morphisms is given by stacking diagrams

$$f \circ g = \substack{f \\ g}$$

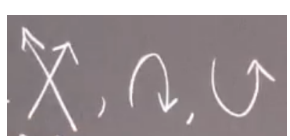
When a "bubble" occurs in such a stacking, we replace it with δ:



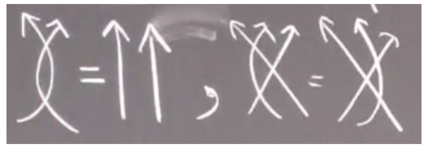
The tensor product of morphisms is given by left-right juxtaposition.

$$f \otimes g = fg$$

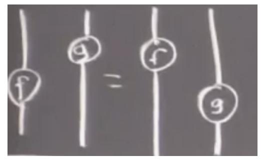
This category has a **monoidal presentation** given by generating objects $\wedge, \vee$, generating morphisms (here we omit half the arrows because the strands are oriented):



and relations which take some work. We start with the braid relations:

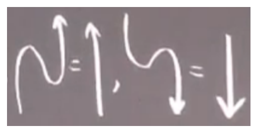


Note that the relation



follows from monoidal axioms (the **interchange law**).

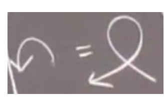
The next relation says that $\vee$ is a right dual to $\wedge$:



We define the shorthand

together with the relation that it be invertible with inverse

Finally, in order to make sense of the "bubble = δ" relation, the top cap must be seen as shorthand for:



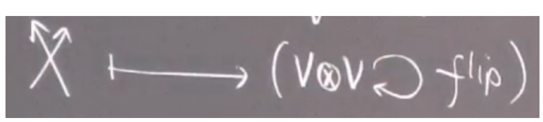
Where does this come from? It really comes from Schur-Weyl duality. This category $\mathrm{OB}(n)$ is the free rigid symmetric monoidal category generated by one object ($\wedge$) of dimension (in the sense of monoidal categories) δ.

Another example of a rigid symmetric monoidal category is the category of finite dimensonal rational representations

$$\mathrm{Rep}\, GL_n$$

is generated by the canonical representation V of column vectors, so the universal property induces

$$\mathrm{OB}(n) \to \mathrm{Rep}\, GL_n$$
$$\wedge \mapsto V$$



We will denote by Kar the Karoubi envelope (extend additively and by images of idempotents). The above functor extends to

$$\mathrm{Kar}(\mathrm{OB}(n)) \to \mathrm{Rep}\, GL_n$$

Theorem 0.1 *(& definition) (Deligne)*
For $\mathbf{k} = \mathbb{C}$,

$$\underline{\mathrm{Rep}} GL_\delta := \mathrm{Kar}(\mathrm{OB}(\delta)).$$

- *It's semisimple* $\iff \delta \notin \mathbb{Z}$,
- *If* $\delta = \pm n$, *with* $n \in \mathbb{N}$, *then the semisimplification (quotient by tensor ideal of negligible morphisms* $\mathcal{N}$*) is equivalent to the representations of* GL,

$$\underline{\mathrm{Rep}} GL_\delta / \mathcal{N} \approx \mathrm{Rep}\, GL_n,$$

and the crossing is assigned to $\pm$ flip *depending on whether* $\delta = \pm n$ *(here* flip *denotes the standard tensor flip).*

This dependence on sign is indicative of a superalgebra!

Definition The category SVec consists of finite-dimensional super vector spaces

$$V = V_{\overline{0}} \oplus V_{\overline{1}},$$

This is a symmetric monoidal category, with braiding given by

$$V \otimes W \to W \otimes V,$$

$$v \otimes w \mapsto (-1)^{\overline{v}\,\overline{w}} w \otimes v.$$

Rigidity of this category gives

$$\dim V = \dim V_{\overline{0}} - \dim V_{\overline{1}}.$$

Recall that GL_n is a group scheme, given by a functor

$$(\text{ commutative } \mathbf{k}\text{-algebras}) \to (\text{groups})$$

$$GL_n(A) = Mat_n(A)^{\times}.$$

We define the **general linear supergroup** $GL_{m|n}$, a group superscheme:

$$(\text{ commutative } \mathbf{k}\text{-superalgebras}) \to (\text{groups})$$

$$GL_{m|n}(A) = (Mat_{m|n}(A)_{\overline{0}})^{\times}.$$

(A **commutative superalgebra** is an algebra $A = A_{\overline{0}} \oplus A_{\overline{1}}$ with the property that $ab = (-1)^{\overline{a}\,\overline{b}} ba$.)
Here $Mat_{m|n}(A)_{\overline{0}}$ denotes the set of block $(m+n) \times (m+n)$ matrices with homogeneous entries as specified:

$$\left(\begin{array}{c|c} A_{\overline{0}} & A_{\overline{1}} \\ \hline A_{\overline{1}} & A_{\overline{0}} \end{array}\right)$$

(where the upper left block is $m \times m$, etc).
The category $\mathrm{Rep}(GL_{m|n})$ has the canonical representation $V = V_{\overline{0}} \oplus V_{\overline{1}}$ with $\dim V_{\overline{0}} = m$ and $\dim V_{\overline{1}} = n$.
We define the subcategory

$$\mathrm{Rep}(GL_{m|n}, z)$$

of representations for which

$$z := \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & -I_n \end{array}\right)$$

acts on v as $(-1)^{\overline{v}}$.

Remark Now we can restate the characterization of $\underline{\text{Rep}}GL_\delta/\mathcal{N}$ in Theorem 0.1 as

$$\underline{\text{Rep}}GL_\delta/\mathcal{N} \approx \begin{cases} \text{Rep}(GL_{n|0}, z) & \text{if } \delta = n, \\ \text{Rep}(GL_{0|n}, z) & \text{if } \delta = -n, \end{cases}$$

and the sign that occurs in the assignment of the crossing becomes more natural.

Theorem 0.2 *(Comes, Coulembier)*
Let $\mathbf{k} = \mathbb{C}$*,* $\delta \in \mathbb{Z}$*. The lattice of tensor ideals in* $\underline{\text{Rep}}GL_\delta$ *is in one-to-one correspondence with* $\mathbb{N}$*.*

$$\mathcal{N} \to I_0,$$

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

where I_k *is the kernel of tensor functor*

$$\underline{\text{Rep}}GL_\delta \mapsto \text{Rep}(GL_{m|n}, z)$$

with

$$m - n = \delta, \min(m, n) = k.$$

We've defined $GL_{m|n}$; we can define other supergroups, such as $OSp_{m|2n}$: given for $V = V_{\bar{0}} \oplus V_{\oplus 1}$ of respective dimensions m and $2n$ with $(\cdot, \cdot)$ a non-degenerate even supersymmetric bilinear form,

$$OSp_{m|2n} = \{g \in GL_{m|2n}(A) \mid (gv, gw) = (v, w) \forall v, w \in V \otimes_{\mathbf{k}} A\}.$$

This unifies symplectic and orthogonal groups.

There are two more families of groups in superalgebra that don't have analogues in ordinary group theory. For $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with respective dimensions n and n, and $(\cdot, \cdot)$ a non-degenerate odd supersymmetric form, we define

$$P_n = \{g \in GL_{n|n}(A) \mid (gv, gw) = (v, w) \forall v, w \in V \otimes_{\mathbf{k}} A\},$$

and if J is an odd involution on the same V,

$$Q_n(A) = \{g \in GL_{n|n}(A) \mid gJ = Jg\}.$$

In coordinates,

$$Q_n(A) = \left\{ \left(\begin{array}{c|c} X & Y \\ \hline Y & X \end{array} \right) \in GL_{n|n}(A) \right\}.$$

Remark Schur-Weyl duality for projective representations of symmetric groups lands in the Q_n family.

These four groups

$$GL_{m|n}, OSp_{m|2n}, P_n, Q_n$$

make the **4 families of classical supergroups**. Over the complex numbers they all have associated Lie superalgebras.

All four families have **Deligne categories**:

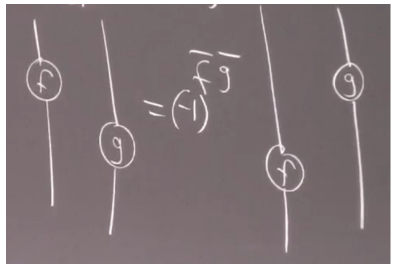
For $OSp_{m|2n}$, we have

$$\underline{\text{Rep}}OSp_\delta,$$

for $\delta = m - 2n$. Here we have the usual (as apposed to oriented) Brauer category.

Similarly for P_n we have $\underline{\text{Rep}}P$ and for Q_n we have $\underline{\text{Rep}}Q$. Importantly, these latter two Deligne categories must be viewed as **monoidal supercategories** (monoidal categories enriched in supervector spaces).

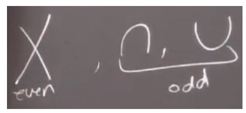
Here instead of the interchange law, we have the **super interchange law**



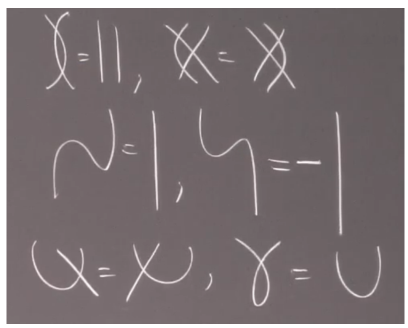
We have

$$\underline{\mathrm{Rep}}P = \mathrm{Kar}(\mathcal{C}),$$

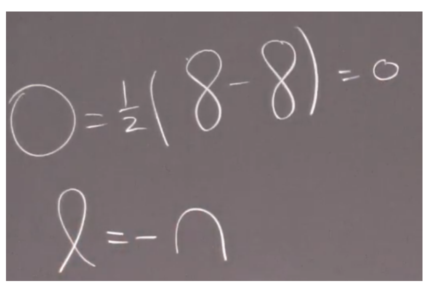
where $\mathcal{C}$ will be a supermonoidal category with a presentation: a generating object is $\bullet$, with morphisms



with relations:

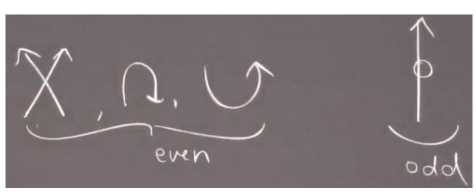


The bubble is forced to be zero by the relations.

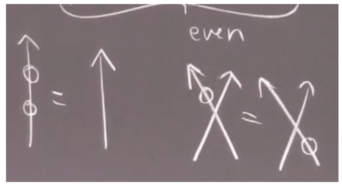


Recall that this is expected by the fact that the dimension of V is 0.

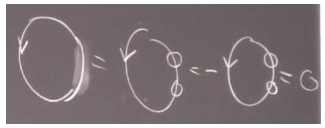
For $\underline{\mathrm{Rep}}Q$, we have a presentation the same as for OB with additional morphisms and relations. The morphisms from OB are all defined to be even, and the new morphism is odd:



and we have new relations



Note that this implies the bubble is zero (using our assumption that $\operatorname{char} \mathbf{k} \neq 2$):



The finite dimensional characters of irreducible representations for all 4 classical supergroup families have been solved (GL: Servanova 1990s, Brundan 2003; Q: Penkov Serganova 1990s, Brundan 2003; OSp: Gruson, Serganova 2007, Ehrig Stroppel 2013; P: 10 author paper 2016) and much work has been done on the category $\mathcal{O}$ for the corresponding Lie superalgebras (GL: Kazhdan-Lusztig conjecture by Brundan 2003 proved by Cheng-Lam-Wang 2014, reproved using rigity theorems of Losev-Webster by Brundan-Losev-Webster 2017; OSp: Bao-Wang 2015; Q: some blocks Brundan-Davidson, 2017; not much has been done on P).

The theory of GL is understood in terms of 2-representations in the sense of Rouquier, using the Kac-Moody 2-categories

$$\mathcal{U}(\mathfrak{sl}_\infty), \mathcal{U}(\widehat{\mathfrak{sl}}_p)$$

built from quiver Hecke algebras.

To understand Q we have super Kac-moody 2-categories

$$\mathcal{U}(\mathfrak{b}_\infty), \mathcal{U}(\mathfrak{a}_{2\ell}^{(2)})$$

built from quiver Hecke superalgebras due to Kang-Kashiwara-Tsuchioka.

For OSp, we have constructions due to Bao-Shan-Wang-Webster.