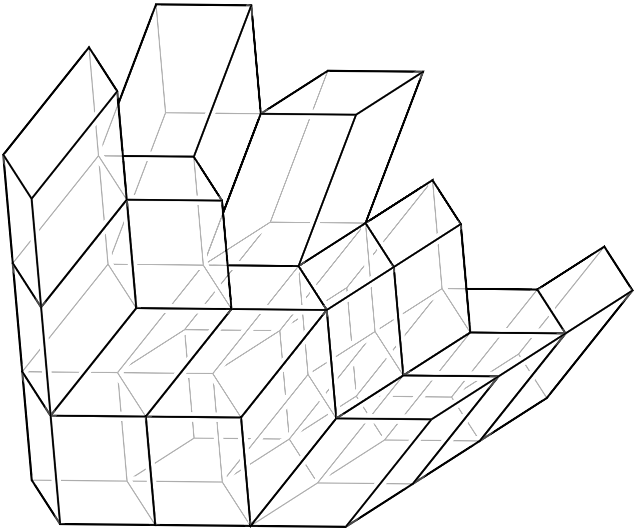
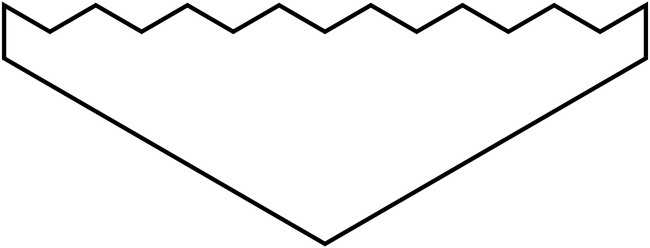
Parallelotope tilings for symmetric groups

Joe Chuang, Hyohe Miyachi and Kai Meng Tan

MSRI, April 2018



2D tilings



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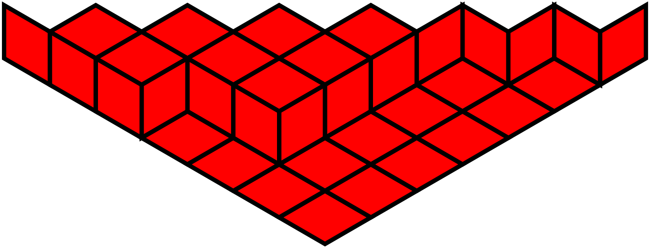

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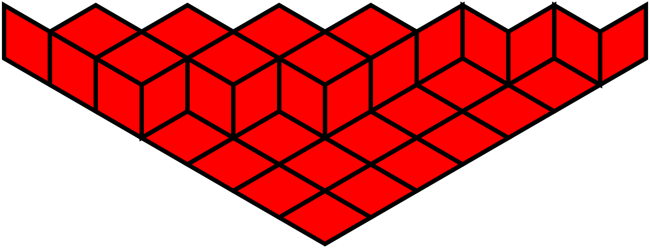

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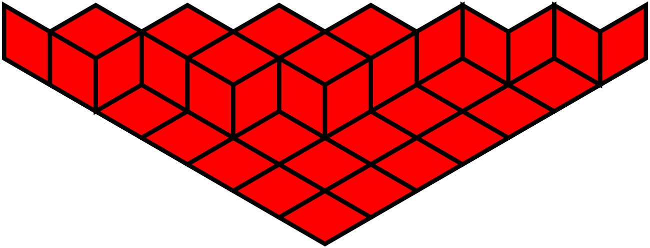

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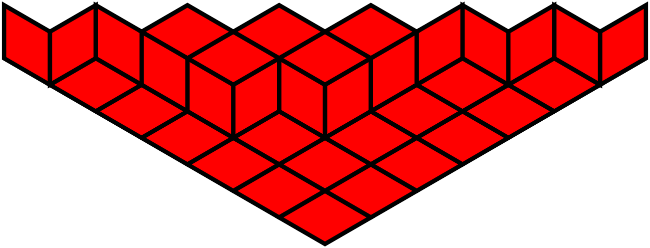

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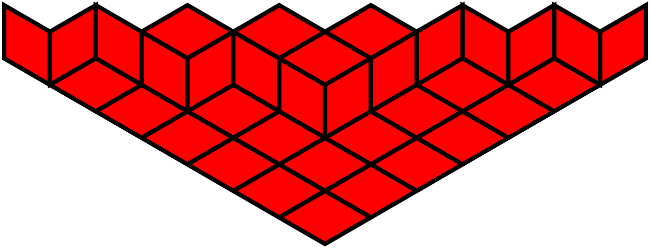

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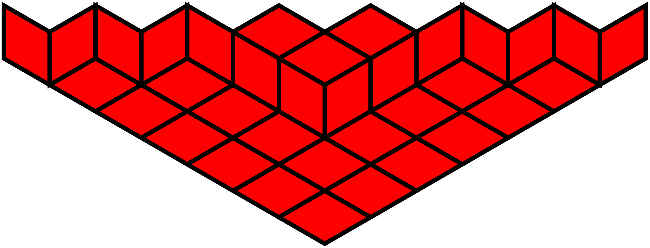

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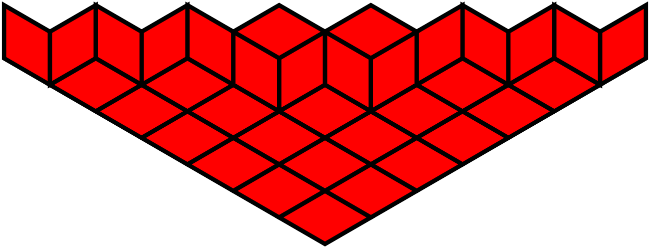

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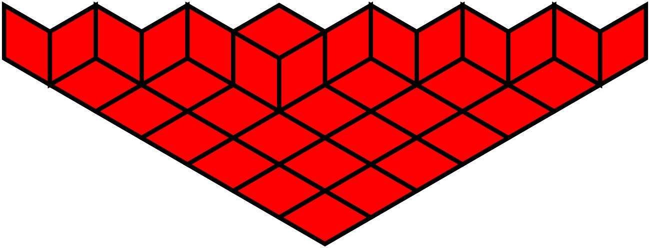

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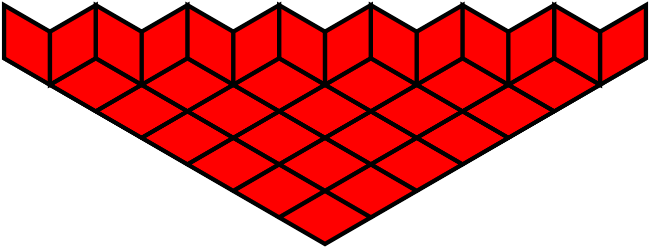

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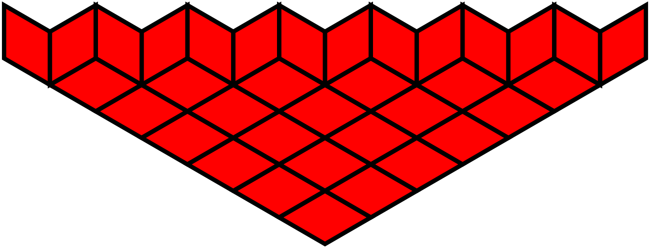

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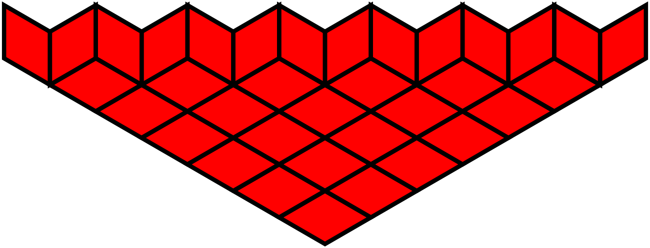

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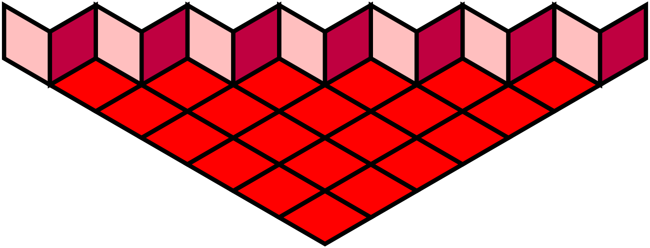

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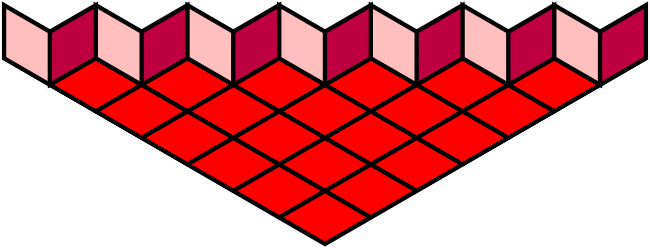

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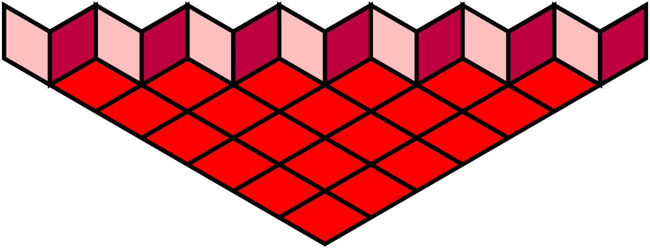

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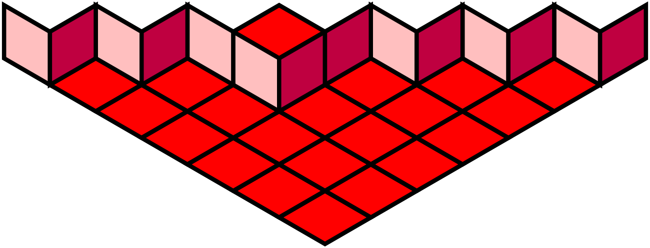

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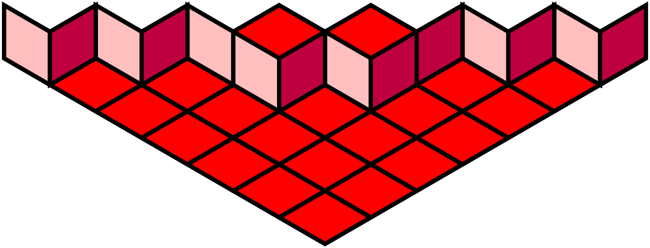

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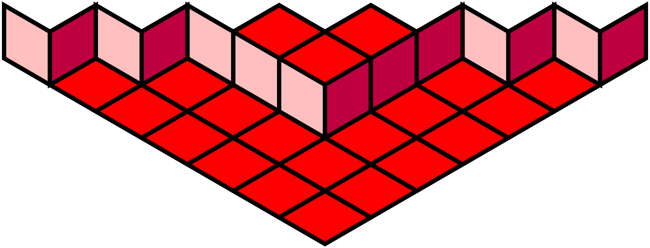

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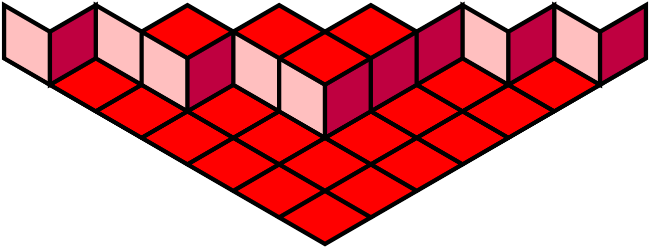

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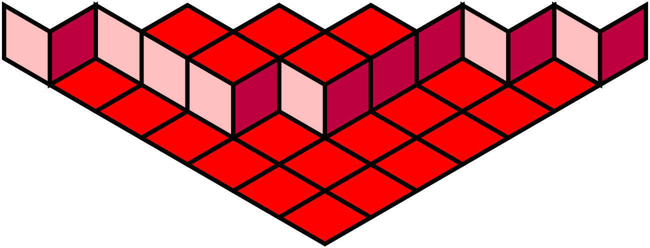

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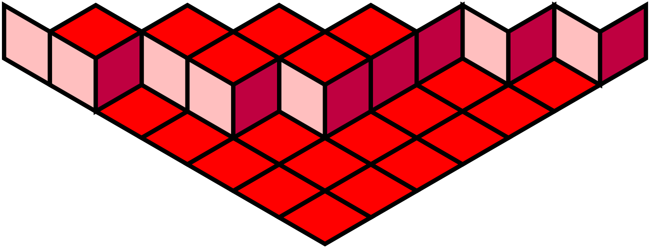

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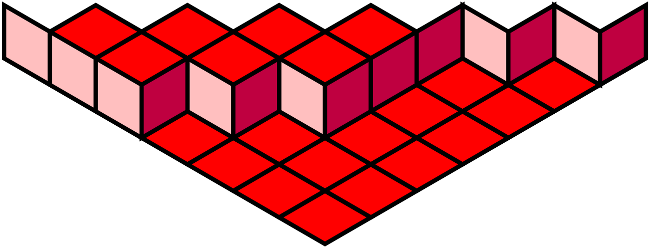

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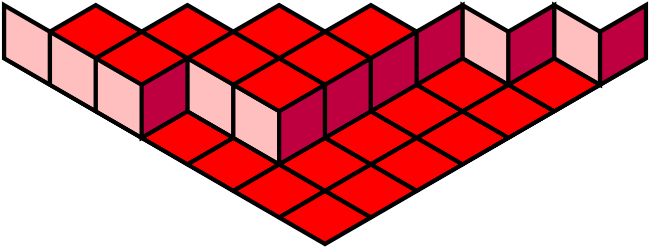

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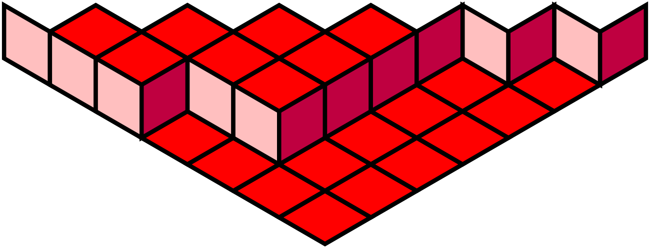

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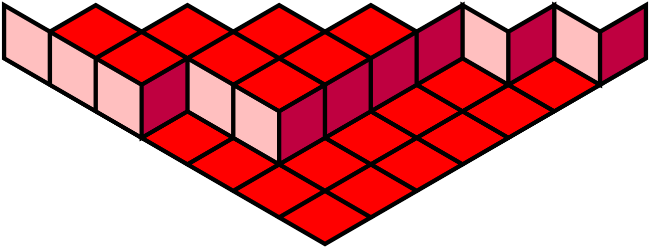

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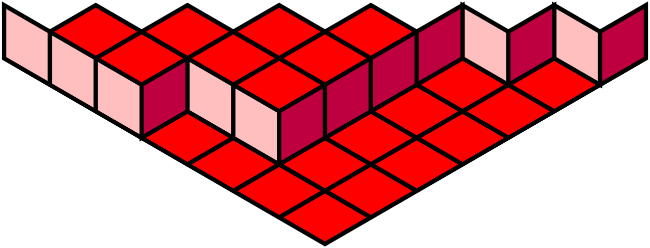

These tilings appear in the representation theory of the symmetric groups (Martin, Peach, Turner).

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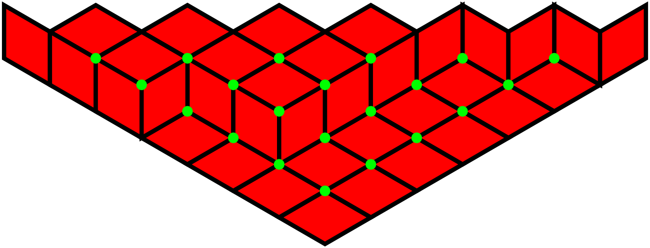
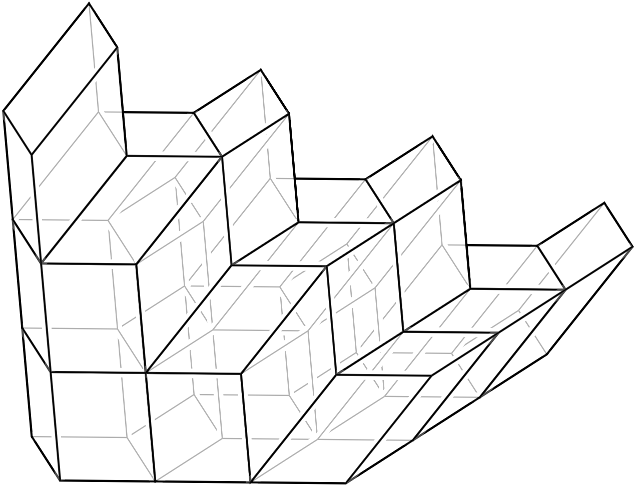


These tilings appear in the representation theory of the symmetric groups (Martin, Peach, Turner). Their 35×21 incidence matrices are submatrices of *decomposition matrices*.

3D tilings



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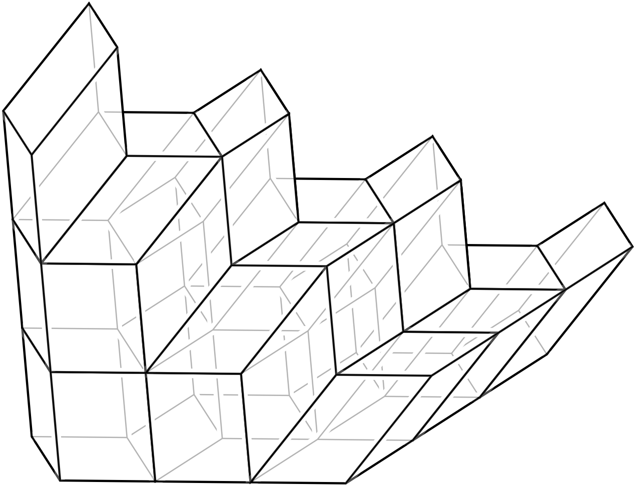

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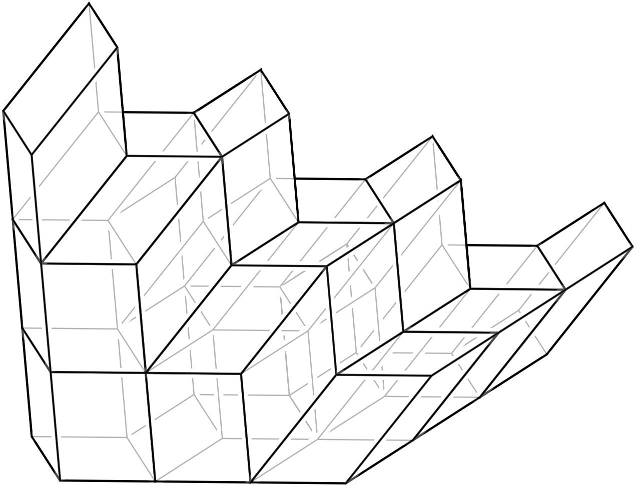

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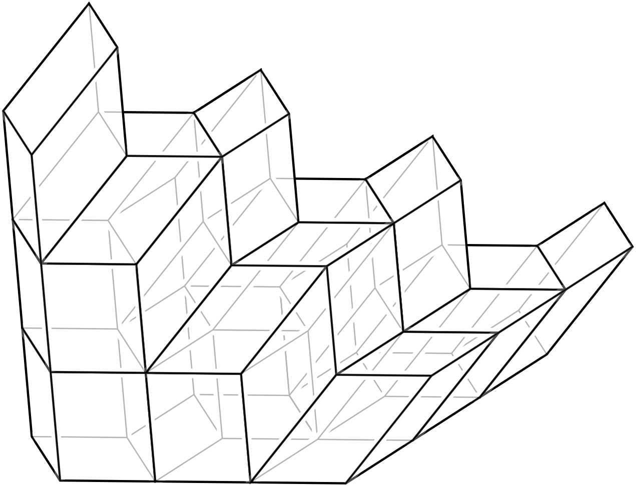

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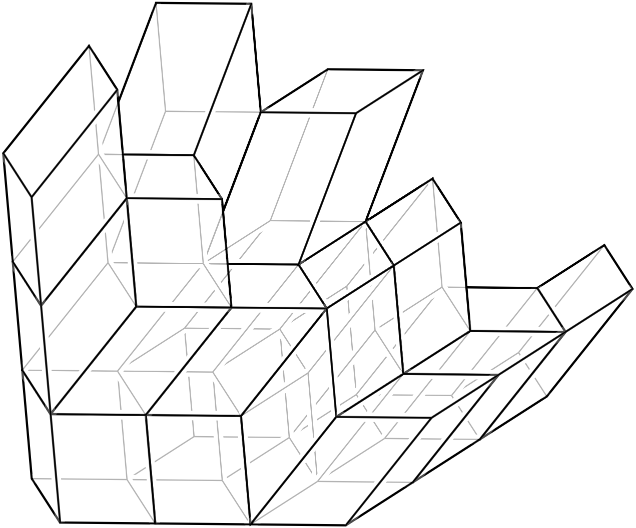

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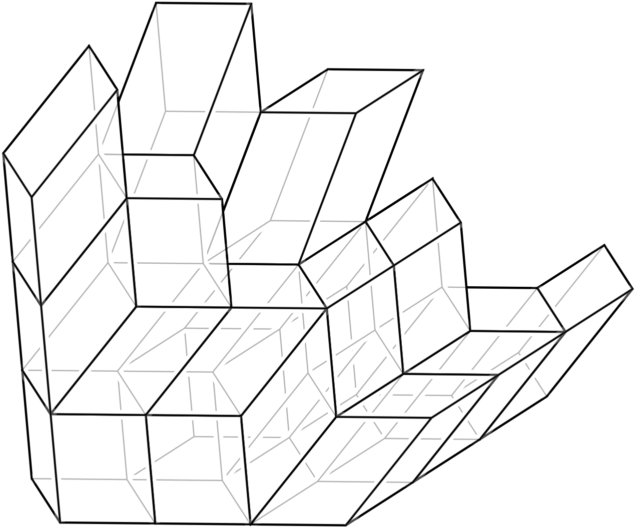

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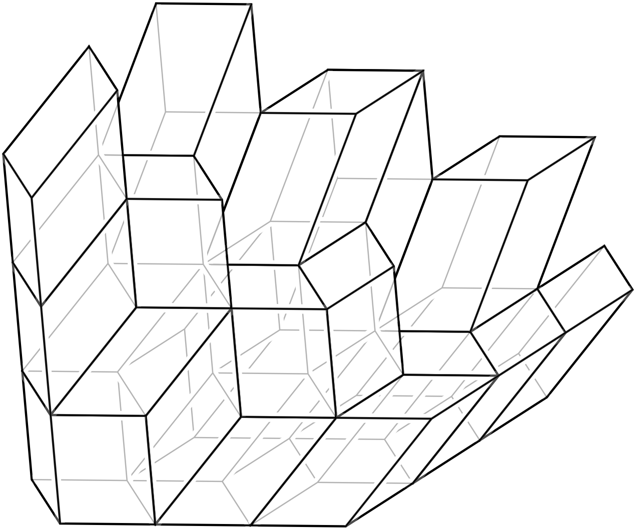

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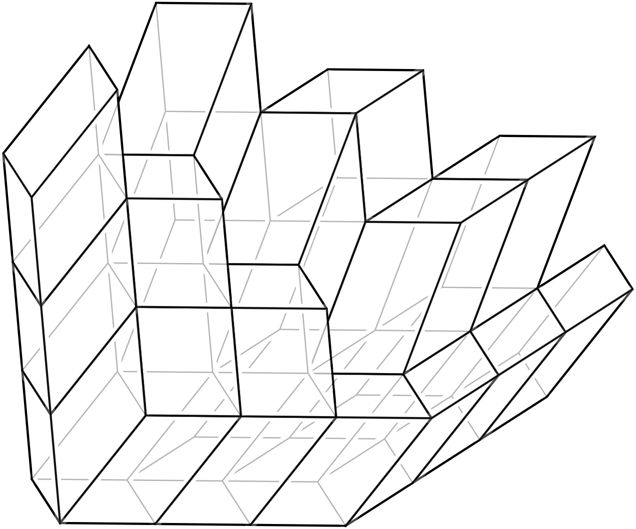

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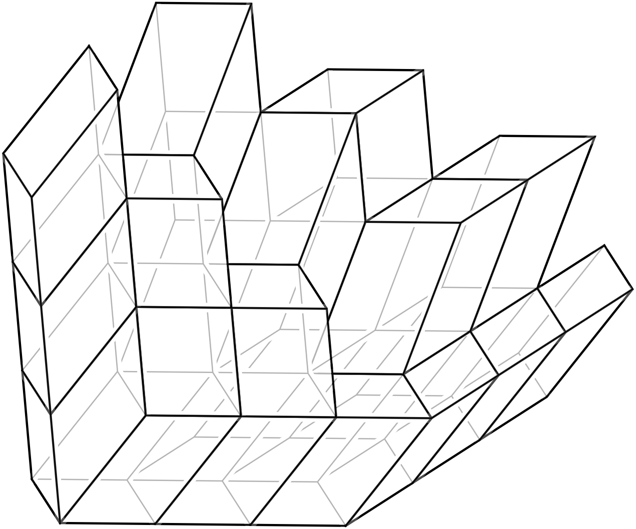

Problem

Determine the dimensions of the simple modules of the symmetric groups $\mathfrak{S}_n$ over a field of characteristic p.

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These can be computed from the *decomposition numbers*

$$[\text{Specht} : \text{simple}].$$

Blocks

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- There are infinitely many such blocks, but only finitely many Morita equivalence classes.
- Any two such blocks are derived equivalent.
- If $w < p$ ($\iff$ abelian defect) then there is one such block Morita equivalent to the principal block of $\mathfrak{S}_p \wr \mathfrak{S}_w$.

Weight 1

Brauer tree

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Brauer tree $(p = 7)$

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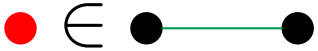
$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \bullet\!-\!\!-\!\!-\!\bullet \\ 0 & \text{otherwise} \end{cases}$$



Weight 1

dual Brauer tree ($p = 7$)

$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \text{ } \\ 0 & \text{otherwise} \end{cases}$$

Weight 1

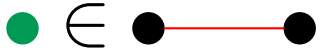
extended dual Brauer tree ($p = 7$)

$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \bullet\!-\!\!-\!\bullet \\ 0 & \text{otherwise} \end{cases}$$

Weight 1

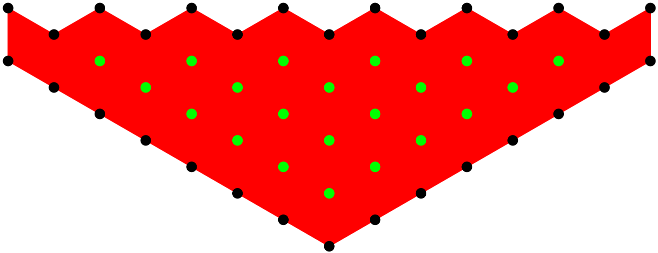
extended dual Brauer tree $(p = 7)$

$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \text{(edge)} \\ 0 & \text{otherwise} \end{cases}$$



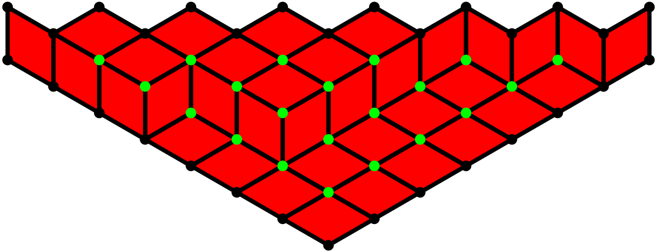
For every block B of weight 1, we have $B \cong e\Gamma e$, where Γ is the Brauer tree algebra of an infinite line.

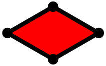
Weight 2



35 Specht modules, 27 simple modules in each 7-block of weight 2.

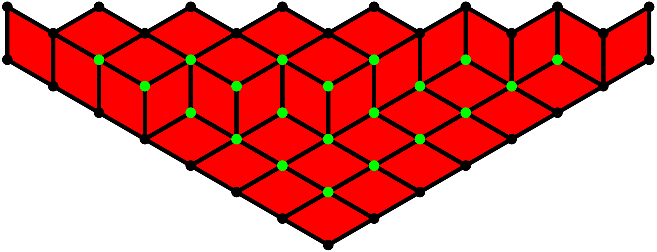
Weight 2

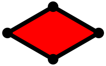


$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \end{cases}$$ 
$$\begin{cases} 0 & \text{otherwise.} \end{cases}$$

$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{64}$.

Weight 2

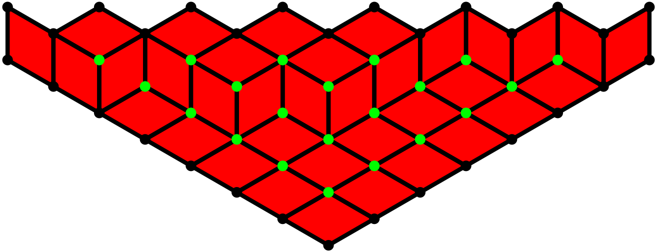


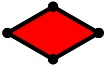
$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \end{cases}$$ 

$$\begin{cases} 0 & \text{otherwise.} \end{cases}$$

$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{72}$.

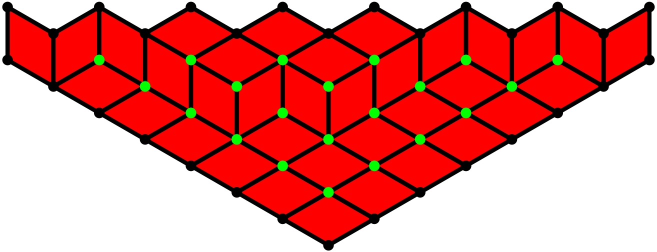
Weight 2

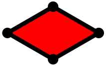


$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \end{cases}$  $\begin{cases} & \\ 0 & \text{otherwise.} \end{cases}$

$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{84}$.

Weight 2

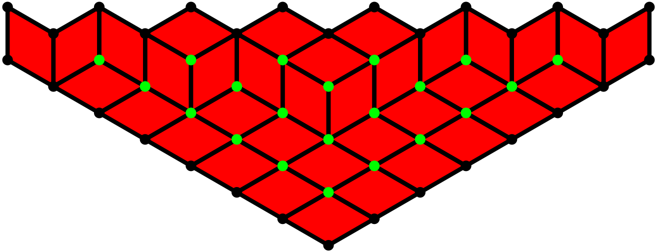


$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \end{cases}$$ 

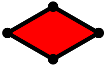
$$\qquad\qquad\qquad\qquad\quad 0 \quad \text{otherwise.}$$

$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{103}$.

Weight 2

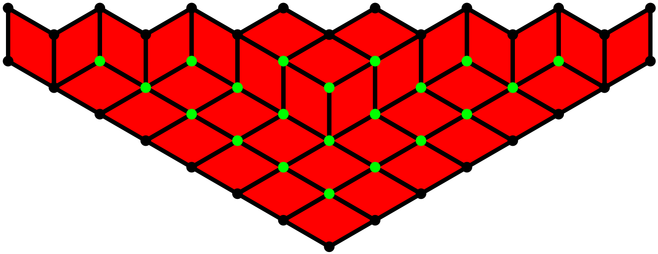


$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \text{(see figure)} \\ 0 & \text{otherwise.} \end{cases}$$

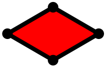


$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{107}$.

Weight 2

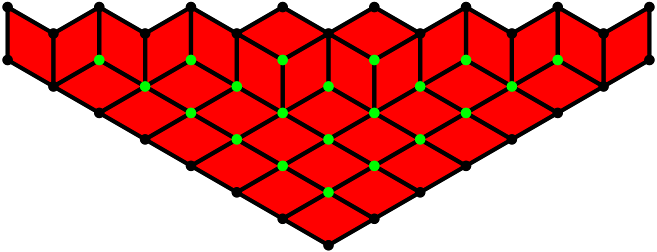


$$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \text{(tile)} \\ 0 & \text{otherwise.} \end{cases}$$



$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{121}$.

Weight 2



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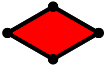
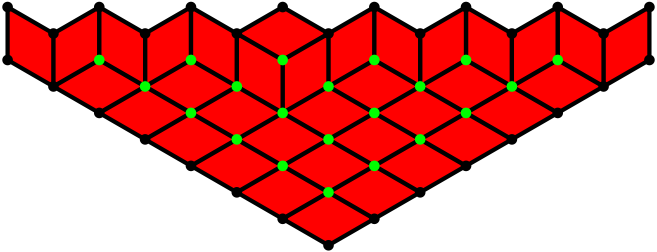


$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{136}$.

Weight 2



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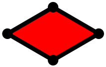
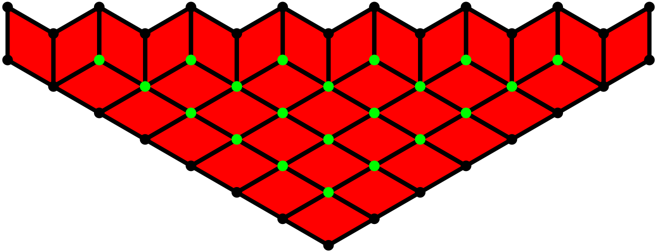
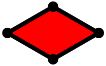


$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{139}$.

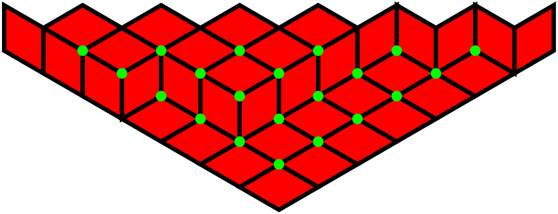
Weight 2



$[\text{Specht} : \text{simple}] = \begin{cases} 1 & \text{if } \bullet \in \end{cases}$ 
$\qquad\qquad\qquad\qquad\quad 0 \quad \text{otherwise.}$

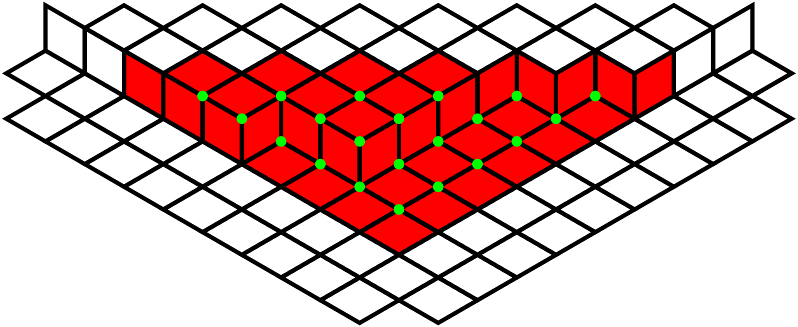
$35/35$ Specht modules, $21/27$ simple modules in certain weight 2 block of $\mathbb{F}_7\mathfrak{S}_{140}$.

Weight 2: Rhombal algebras



The tiling T_B associated to a block B

Weight 2: Rhombal algebras



The tiling T_B associated to a block B extends to a tiling $\hat{T}_B$ of the whole plane, and $eBe \cong e\Gamma(\hat{T})e$, where $\Gamma(\hat{T}_B)$ is one of Peach's *rhombal algebras*.

Cubist algebras

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$$T\colon \qquad \mathbb{R}^d = \bigcup_{\gamma} P_\gamma,$$

by parallelotopes P_γ, each generated by a basis of Λ^d, and in which any two P_γ's are disjoint or intersect in a common face.

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- $\overline{\lambda}_1\alpha_1\alpha_1' + \cdots + \overline{\lambda}_s\alpha_s\alpha_s' = 0$ if $\lambda_1 v(\alpha_1) + \cdots + \lambda_s v(\alpha_s) = 0$ in Λ^d.

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Turner: $U_T := V_T^!$ is a locally finite-dimensional algebra with many good homological properties.

q-decomposition numbers

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with equality known to hold in many cases when λ and μ have p-weight $w < p$.

e-divisible rimhooks

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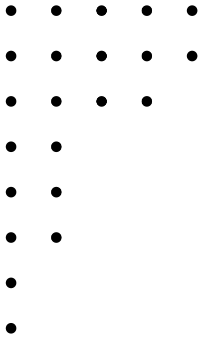
Then wrap on

$$\tilde{H}^{\mathrm{last}}, \ldots, \tilde{H}^1,$$

where $\tilde{H}^i$ is minimally strictly north-east of H^i, to obtain $[\lambda_H]$.

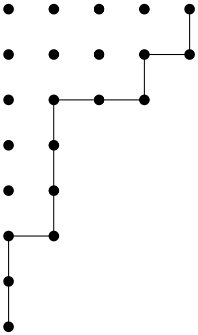
e-divisible rimhooks: example

$\lambda = (5^2, 4, 2^3, 1^2)$



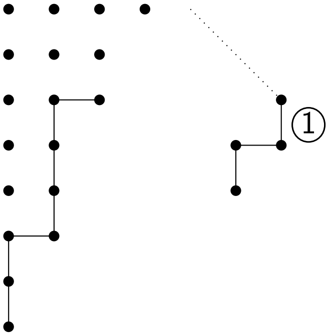
e-divisible rimhooks: example

$\lambda = (5^2, 4, 2^3, 1^2) \qquad H \in \mathrm{Hook}_4(\lambda)$



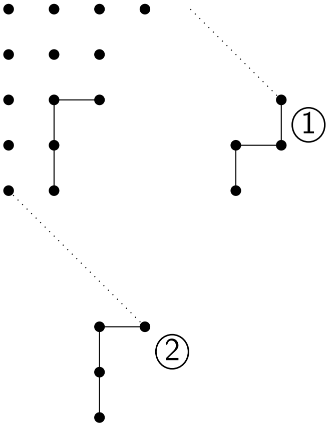
e-divisible rimhooks: example

$\lambda = (5^2, 4, 2^3, 1^2)$ Unwrap H^1



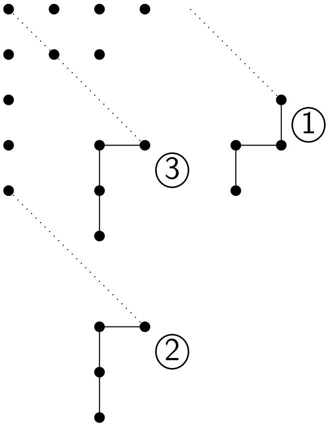
e-divisible rimhooks: example

$\lambda = (5^2, 4, 2^3, 1^2)$ Unwrap H^2



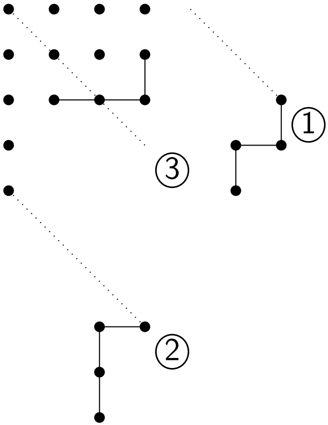
e-divisible rimhooks: example

$\lambda = (5^2, 4, 2^3, 1^2)$ Unwrap H^3



e-divisible rimhooks: example

$\lambda = (5^2, 4, 2^3, 1^2)$ Wrap $\tilde{H}^3$



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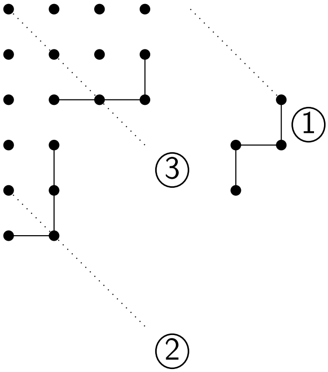

e-divisible rimhooks: example

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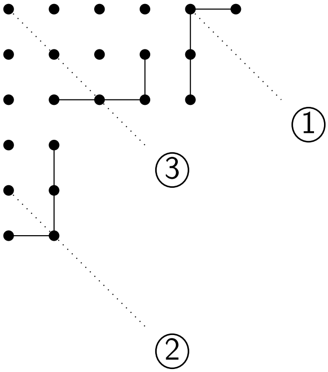
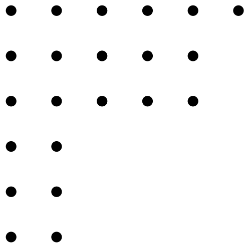


e-divisible rimhooks: example

$\lambda = (5^2, 4, 2^3, 1^2) \qquad \lambda_H = (6, 5^2, 2^3)$



Theorem (CMT)

Let λ *be* generic *(to be defined later). Then for all* $H \in \mathrm{Hook}_e(\lambda)$, λ_H *is well-defined, and*

$$d_{\lambda,\lambda_H}(q) = q.$$

Denote by B_{gen} the subset of generic partitions in a block B of e-weight w. Then

$$\frac{|B_{\mathrm{gen}}|}{|B|} \to 1 \quad \text{as} \quad e \to \infty \quad (w \text{ fixed}).$$

Theorem (CMT)

There exists a map $v\colon B_{\mathrm{gen}} \to \mathbb{Z}^w$ *such that for all* $\lambda, \mu \in B_{\mathrm{gen}}$, *we have* $d_{\lambda\mu}(q) = q^{|\Omega|}$ *if* $v(\mu) = v(\lambda) + \sum_{H \in \Omega}(v(\lambda_H) - v(\lambda))$ *for some* $\Omega \subseteq \mathrm{Hook}_e(\lambda)$,

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Moreover the solid w*-parallelotopes*

$$\Pi(\lambda) = \{v(\lambda) + \sum_{H \in \mathrm{Hook}_e(\lambda)} a_H(v(\lambda_H) - v(\lambda)) \colon a_H \in [0, 1]\}$$

are the w*-cells of a connected, simply connected polytopal complex in* $\mathbb{R}^w$.

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Define v_i to be the width of H_i^{last}, or one less than the width if the NE-most box of H_i^{last} is the last box in its row of $[\lambda]$.

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And $v : B_{\mathrm{gen}} \to \mathbb{Z}^w$ is given by $v(H) = (v_1, \ldots, v_w)$.

Example: $e = 3$, $w = 2$, empty 3-core.

None of the partitions in this block are generic, but we can still check...

λ
(6)
$(5, 1)$
$(4, 1^2)$
(3^2)
$(3, 2, 1)$
$(3, 1^3)$
(2^3)
$(2, 1^4)$
(1^6)

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λ	$v(\lambda)$	e_1^λ	e_2^λ
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$(5,1)$	$(2,2)$	e_1	$e_2 - e_1$
$(4,1^2)$	$(1,2)$	e_1	$e_2 - e_1$
(3^2)	$(2,1)$	e_1	e_2
$(3,2,1)$	$(1,1)$	e_1	e_2
$(3,1^3)$	$(0,2)$	$e_1 - e_2$	e_2
(2^3)	$(1,0)$	e_1	e_2
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(6)	$(3,2)$	e_1	$e_2 - e_1$					
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(3^2)	$(2,1)$	e_1	e_2					
$(3,2,1)$	$(1,1)$	e_1	e_2					
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q-decomposition matrix (3-regular μ), predicted:

λ	$v(\lambda)$	e_1^λ	e_2^λ	(6) $(3,2)$	$(5,1)$ $(2,2)$	$(4,1^2)$ $(1,2)$	(3^2) $(2,1)$	$(3,2,1)$ $(1,1)$
(6)	$(3,2)$	e_1	$e_2 - e_1$	1				
$(5,1)$	$(2,2)$	e_1	$e_2 - e_1$	q	1			
$(4,1^2)$	$(1,2)$	e_1	$e_2 - e_1$		q	1		
(3^2)	$(2,1)$	e_1	e_2	q^2	q		1	
$(3,2,1)$	$(1,1)$	e_1	e_2		q^2	q	q	1
$(3,1^3)$	$(0,2)$	$e_1 - e_2$	e_2			q^2		q
(2^3)	$(1,0)$	e_1	e_2				q^2	q
$(2,1^4)$	$(0,1)$	$e_1 - e_2$	e_2					q^2
(1^6)	$(0,0)$	$e_1 - e_2$	e_2					

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q-decomposition matrix (3-regular μ), actual:

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(3^2)	$(2,1)$	e_1	e_2	~~q^2~~	q		1	
$(3,2,1)$	$(1,1)$	e_1	e_2	q	q^2	q	q	1
$(3,1^3)$	$(0,2)$	$e_1 - e_2$	e_2			q^2		q
(2^3)	$(1,0)$	e_1	e_2	q^2			~~q^2~~	q
$(2,1^4)$	$(0,1)$	$e_1 - e_2$	e_2				q	q^2
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