Finite groups of Lie type and (q,t)-polynomials

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Based on past/ongoing work with

Chuang, Craven, Bonnafé, Shan, Varagnolo, Vasserot, Dudas

Modular representations and local group theory

G finite group, k char $\ell > 0$ field

$$\text{Rep } G \text{ over } k \overset{?}{\longleftrightarrow} \{\text{Rep } N_G(Q)\text{'s}\}_{1 \neq Q \leq G \ \ell\text{-subgroup}}$$

* $H^*(G, k)$ ✓
* # irred. rep (in a block) : Alperin's Conj } + abelian defect : Broué's Conj on derived cat.
* Dim irred. rep, dec. matrix ?

q- dec. matrices : S_n, $GL_n(\mathbb{F}_{p^r})$ $(\ell >> n)$

Other finite groups of Lie type $(U_n, Sp_{2n}, \dots)$?

Resolution of cohomology

Quillen: $H^*(GL_n(\mathbb{F}_{p^r}), k) = \left(S(k^n) \otimes \Lambda(k^n)\right)^{S_n}$ $\quad$ $\mathrm{Hilb}^n k^2$ $\quad$ min. resol. of sing.

$\ell \mid p^r - 1,\ \ell > n$ $\quad$ Koszul $\quad \downarrow$

$\left(S(k^n) \otimes S(k^n)\right)^{S_n}$: fcts on k^{2n}/S_n

Thm | $\exists$ defo. of $\mathrm{Hilb}^n k^2$ that is derived equiv. to p$\underline{\text{al}}$ block of $kGL_n(p^r)$

* $(k^\times \times k^\times)$-action $\rightsquigarrow$ (q,t)-dec. matrix
* Connection with double affine Hecke algebras

$\Big\}$ makes sense for all finite groups of Lie type

Rem. $H^*(G,k)$: nbh of trivial rep.

Challenge: move away from trivial rep.

Decomposition matrices for finite unitary groups (Dudas-R)

$G = U_n(p^r) = GL_n(-p^r) = GL_n(\overline{\mathbb{F}}_{p^r})^{((a_{ij})\mapsto {}^t(a_{ij}^{p^r})^{-1})}$ $\qquad$ $kG\text{-mod}$

$\qquad \wr$

Jordan decomposition (Lusztig, Broué, Bonnafé-Dat-R): $\bigoplus_{s\in (G^*)^{F^*}_{\ell'}/\sim} kC_{G^*}(s)^{*F}\text{-Unip}$

$\mathrm{Irr}(kU_n(p^r)\text{-Unip}) \xrightarrow{\sim}$ Partitions of n

$\qquad\qquad\cup \qquad\qquad\qquad\qquad \cup$

principal series $\xrightarrow{\sim}$ empty 2-core

Conj | unipotent dec. mat $U_n(p^r) = D_n(q,t)_{/q=t=1}$, $\ell \mid p^r+1$, $\ell >> n$

$D_n(q,t)$ defined via Macdonald polynomials

true for $n \leq 9$ (based on Dudas-Malle)

Version for any d, $\ell \mid (p^r)^d - 1$. New (two var.) bases of Fock space (... stable bases)

Conj | $kU_n(p^r)\text{-Unip} \simeq$ deformation of heart of "explicit" t-structure on $\mathrm{Hilb}^n k^2$

$\mathcal{D}_8(q,t)_{((8),(1^8))}$ $\qquad\qquad$ $\mathcal{D}_8(1,1)_{((8),(1^8))}=0$

$\parallel$

-t^9*q^31 - t^11*q^27 - t^9*q^29 + t^7*q^31 - t^21*q^15 - t^19*q^17 - t^17*q^19 - t^15*q^21 - 2*t^13*q^23 - 2*t^11*q^25 - t^9*q^27 + 2*t^7*q^29 - t^21*q^13 - t^19*q^15 - 2*t^17*q^17 - 2*t^15*q^19 - 3*t^13*q^21 - 2*t^11*q^23 + t^9*q^25 + 4*t^7*q^27 - t^5*q^29 + t^19*q^13 - t^13*q^19 + 2*t^11*q^21 + 5*t^9*q^23 + 4*t^7*q^25 - 2*t^5*q^27 + t^19*q^11 + 2*t^17*q^13 + 4*t^15*q^15 + 3*t^13*q^17 + 7*t^11*q^19 + 7*t^9*q^21 + 3*t^7*q^23 - 4*t^5*q^25 + t^17*q^11 + 3*t^15*q^13 + 4*t^13*q^15 + 7*t^11*q^17 + 5*t^9*q^19 - 2*t^7*q^21 - 5*t^5*q^23 + t^3*q^25 + 2*t^11*q^15 - t^9*q^17 - 6*t^7*q^19 - 5*t^5*q^21 + t^3*q^23 - t^15*q^9 - 2*t^13*q^11 - 3*t^11*q^13 - 5*t^9*q^15 - 8*t^7*q^17 - 3*t^5*q^19 + 2*t^3*q^21 - t^13*q^9 - 3*t^11*q^11 - 4*t^9*q^13 - 6*t^7*q^15 + 2*t^3*q^19 - t^11*q^9 - t^9*q^11 - 2*t^7*q^13 + 2*t^5*q^15 + 2*t^3*q^17 + t^9*q^9 + 3*t^5*q^13 + t^3*q^15 + t^9*q^7 + t^7*q^9 + 2*t^5*q^11 + t^3*q^13 + t^5*q^9

Degeneration and genericity

Degeneration kG-Unip comes from degeneration to the shifted normal cone

$$
\begin{tikzcd}
{[\mathrm{Spec}(kT\text{-Unip})/W]} \arrow[r, hook] \arrow[dd, leftrightarrow, "\text{derived}\ \sim", "\text{Brou\'e?}"'] \arrow[dr, bend left, "t\neq 0"'] & k\otimes[\mathbb{A}_X/W] & & k\otimes[\mathbb{A}_{X\oplus X[1]}/W] \arrow[ll] \arrow[dll, bend left, "t=0"] \arrow[dd, leftrightarrow, "\text{derived}\ \sim"', "\text{Koszul}"] \\
 & k\otimes V_t \arrow[u] & & \\
kG\text{-Unip} \arrow[r, bend right, "t\neq 0"', "?"] & k\otimes C_t \arrow[u, leftrightarrow, "\sim\ \text{derived}"] & kG(X)\text{-Unip}_d \arrow[l, "t=0"] \arrow[r, leftrightarrow, "\sim", "\text{der.}"'] & k\otimes[\mathbb{A}_{X\oplus X^\vee}/W]
\end{tikzcd}
$$

Here: $G = GL_n(\pm p^r)$

$\ell \mid \pm p^r - 1$, $\ell > n$,

$\mathbb{T}$ max torus $/\mathbb{F}_{p^r}$

$T = \mathbb{T}(\mathbb{F}_{p^r})$

W: Weyl group

$X = X(\mathbb{T})$

General: W cyclotomic Weyl group

$\underbrace{[\mathbb{A}_{X\oplus X^\vee}/W], \text{perv.}}_{\text{ab. cat } G(X)\text{-Unip}_d}$ function: depend only on Lie type and d min., $\ell \mid (p^r)^d - 1$

defined over $\mathbb{Z}[\sqrt[d]{1}]$

$$\bigoplus_{n\geq 0} K_{\mathbb{G}_m}(k\,GL_n(p^r)\text{-Unip}) \longleftarrow \bigoplus_{n\geq 0} K_{\mathbb{G}_m\times\mathbb{G}_m}(\mathbb{Q}\,GL_n(X)\text{-Unip}_d)$$

$U_q(\hat{\mathfrak{sl}}_d)$ char $\ell > 0$ $\qquad$ $U_q(\hat{\mathfrak{sl}}_d)$ further specialization

Cherednik algebras and Spets

W Weyl group $\rightsquigarrow H_{t,c}(W)$ rational Cherednik algebra $\left(H_{0,0}(W) = \mathbb{C}[V \oplus V^*] \rtimes W\right)$

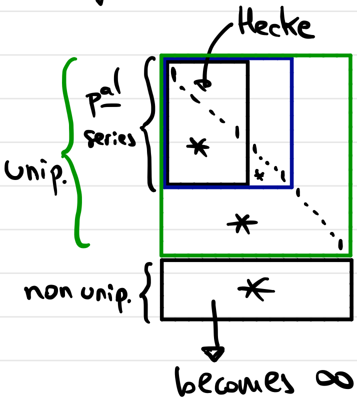
Hope 1: $\exists\ \mathbb{Z}[X, b_1, b_2, \dots]$-algebra $\mathrm{Unip}(W)$ with

$$\mathrm{Unip}(W)|_{X=0,\, b=0} \longrightarrow H_{t=0,\, c=1}(W) \sim \text{Calogero-Moser}$$

$$\mathrm{Unip}(W)|_{X=\sqrt[d]{1},\, b=0} \longrightarrow H_{t=1,\, c=\frac{1}{d}}(W) \quad \left(\sim \mathrm{Schur}_{\sqrt[d]{1}}(n) \text{ for } W = S_n\right)$$

Given ℓ, p^r, $\quad k \otimes \mathrm{Unip}(W)|_{X=p^r,\, b_i = \delta_{i,m}} \simeq k\,\mathbb{G}(p^r)\text{-Unip}$

$m = \ell^a \,\|\, (p^r)^d - 1$



Hope 2: Makes sense for W complex refl. group ($\rightsquigarrow$ Spets)