Deligne–Lusztig Induction and Almost Characters

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Today: Work in progress.

1 Background

Let G be a finite group, and denote by

$$\mathrm{Class}(G)$$

the space of class functions on G over $\mathbb{C}$ (inner product space). This space has an orthonormal basis given by the set of irreducible characters $\mathrm{Irr}(G) \subset \mathrm{Class}(G)$. For any subgroup $L \leq G$ we have associated induction and restriction maps

$$\mathrm{Ind}_L^G : \mathrm{Class}(L) \to \mathrm{Class}(G),$$
$$\mathrm{Res}_L^G : \mathrm{Class}(G) \to \mathrm{Class}(L).$$

Problem: If $\rho \in \mathrm{Irr}(L)$ and $\chi \in \mathrm{Irr}(G)$, what is the multiplicity $\langle \chi, \mathrm{Ind}_L^G(\rho) \rangle$ of χ in $\mathrm{Ind}_L^G(\rho)$?

Example: If $G = \mathfrak{S}_n$ is the symmetric group then the irreducible characters $\mathrm{Irr}(G) = \{\chi_\lambda \mid \lambda \vdash n\}$ are parameterised by partitions of n. If $L = \mathfrak{S}_a \times \mathfrak{S}_b$ is a Young subgroup, with $a + b = n$, then we have the multiplicity

$$\langle \chi_\lambda, \mathrm{Ind}_L^G(\chi_\mu \boxtimes \chi_\nu) \rangle = c_{\mu\nu}^\lambda$$

is given by the Littlewood–Richardson coefficient $c_{\mu\nu}^\lambda$. These coefficients are combinatorially computable by the Littlewood–Richardson rule.

More generally, if G is a finite real reflection group and L is a reflection subgroup then one can determine the multiplicities $\langle \chi, \mathrm{Ind}_L^G(\rho) \rangle$ explicitly/combinatorially.

Finite Reductive Groups

We will consider the case where G is a finite reductive group. In particular, we have a connected reductive algebraic group $\mathbb{G}$ and a Steinberg endomorphism $F : G \to \mathbb{G}$ such that $G = \mathbb{G}^F$. In this setting the usual induction/restriction maps are not the right tools. However, in 1976 Deligne–Lusztig defined induction/restriction maps

$$R_{\mathbb{L}\subseteq\mathbb{P}}^{\mathbb{G}} : \mathrm{Class}(L) \to \mathrm{Class}(G),$$

$${}^*R_{\mathbb{L}\subseteq\mathbb{P}}^{\mathbb{G}} : \mathrm{Class}(G) \to \mathrm{Class}(L),$$

for any parabolic subgroup $\mathbb{P} \leqslant \mathbb{G}$ with F-stable Levi complement $\mathbb{L} \leqslant \mathbb{P}$. We note that these maps send irreducible characters to *virtual* characters, i.e., $\mathbb{Z}$-linear combinations of irreducible characters.

Problem: If $\rho \in \mathrm{Irr}(L)$ and $\chi \in \mathrm{Irr}(G)$, what is the multiplicity $\langle \chi, R_{\mathbb{L}\subseteq\mathbb{P}}^{\mathbb{G}}(\rho) \rangle \in \mathbb{Z}$ of χ in $R_{\mathbb{L}\subseteq\mathbb{P}}^{\mathbb{G}}(\rho)$?

Example: If $\mathbb{P}$ is F-stable then

$$R^{\mathbb{G}}_{\mathbb{L}\subseteq\mathbb{P}} = \underbrace{\mathrm{Ind}_P^G \mathrm{Inf}_L^P}_{R_L^G}$$

is simply Harish-Chandra induction (or parabolic induction).

History:

- (Howlett–Lehrer '84): Comparison Theorem for Harish-Chandra induction.
- Lusztig ('85/'88): Complete solution when $\mathbb{L}$ is a torus.
- Asai ('84/'85): case where ρ is unipotent.
- Shoji ('85/'87): generalization of Asai to arbitrary characters when $Z(\mathbb{G})$ is connected.

Why are we interested in this problem?

Conjecture 1.1 (Lusztig) *There is a basis of* $\mathrm{Class}(G)$ *given by characteristic functions of character sheaves. The conjecture specifies the change of basis matrix from this basis to the basis* $\mathrm{Irr}(G)$*, of irreducible characters, up to scalars.*

The way to think about Lusztig's conjecture is inductive through Levi subgroups. The problem $\langle \chi, R^{\mathbb{G}}_{\mathbb{L}\subseteq\mathbb{P}}(\rho)\rangle$ becomes quite relevant here.

2 Harish-Chandra Theory

We say $\chi \in \mathrm{Irr}(G)$ is **cuspidal** if

$$^*R_L^G(\chi) = 0 \text{ for all } L < G.$$

If $L \leqslant G$ is a Levi subgroup and $\delta \in \mathrm{Irr}(L)$ is a cuspidal character then we consider the Harish-Chandra series $\mathcal{E}(G,(L,\delta)) = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi, R_L^G(\delta)\rangle \neq 0\}$. A classic result of Harish-Chandra shows that we obtain a partition

$$\mathrm{Irr}(G) = \bigsqcup_{(L,\delta)/\sim} \mathcal{E}(G,(L,\delta))$$

of the set of irreducible characters. This yields a corresponding direct sum decomposition of the space of class functions

$$\mathrm{Class}(G) = \bigoplus \mathrm{Class}(G,(L,\delta)).$$

Theorem 2.1 (Howlett–Lehrer, Lusztig, Geck) *Let* $W_G(L,\delta)$ *be the stabilizer of* δ *in* $N_G(\mathbb{L})/L$ *then there is an isometry*

$$R_L^G(\delta \mid -) : \mathrm{Class}(W_G(L,\delta)) \to \mathrm{Class}(G,(L,\delta))$$

such that $R_L^G(\delta \mid \rho) \in \mathcal{E}(G,(L,\delta))$ *for any* $\rho \in \mathrm{Irr}(W_G(L,\delta))$ *and*

$$R_L^G(\delta) = \sum_{\rho \in \mathrm{Irr}(W_G(L,\delta))} \rho(1) R_L^G(\delta \mid \rho).$$

Theorem 2.2 *(Comparison Theorem, due to Howlett-Lehrer)*
If $L \leq M$ *then we obtain a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Class}(G,(L,\delta)) & \xleftarrow{R_L^G(\delta\mid -)} & \mathrm{Class}(W_G(L,\delta)) \\
\uparrow{\scriptstyle R_M^G} & & \uparrow{\scriptstyle \mathrm{Ind}_{W_M(L,\delta)}^{W_G(L,\delta)}} \\
\mathrm{Class}(M,(L,\delta)) & \xleftarrow[R_M^G(\delta\mid -)]{} & \mathrm{Class}(W_M(L,\delta))
\end{array}$$

3 Unipotent Characters

Recall that Lusztig has defined the following set of irreducible characters

$$\mathcal{E}(G,1) = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi, R_{\mathbb{T}}^{\mathbb{G}}(1)\rangle \neq 0 \text{ for some } F\text{-stable maximal torus } \mathbb{T} \leq \mathbb{G}\}$$

whose elements are called **unipotent characters**.

Now assume we fix $\mathbb{T}_0 \leq \mathbb{B}_0 \leq \mathbb{G}$ an F-stable maximal torus and Borel subgroup of $\mathbb{G}$ then we can define a map

$$W_{\mathbb{G}} := N_{\mathbb{G}}(\mathbb{T}_0)/\mathbb{T}_0 \ni w \mapsto \mathbb{T}_w \leq \mathbb{G} \text{ an } F\text{-stable maximal torus.}$$

As $\mathbb{T}_0$ is F-stable the endomorphism F induces an automorphism of $W_{\mathbb{G}}$. We set

$$\widetilde{W}_{\mathbb{G}} := W_{\mathbb{G}} \rtimes \langle F \rangle \supseteq W_{\mathbb{G}}F,$$

where $W_{\mathbb{G}}F$ denotes the unique coset of $W_{\mathbb{G}} \lhd \widetilde{W}_{\mathbb{G}}$ containing F. The group $W_{\mathbb{G}}$ acts on the coset $W_{\mathbb{G}}F$ by conjugation and we denote by $\mathrm{Class}(W_{\mathbb{G}}F)$ the $\mathbb{C}$-valued functions which are invariant under this action. This is a straightforward generalisation of the usual notion of class functions.

The irreducible characters of the coset are defined to be the elements of the following set

$$\mathrm{Irr}(W_{\mathbb{G}}F) = \{\mathrm{Res}_{W_{\mathbb{G}}F}^{\widetilde{W}_{\mathbb{G}}}(\widetilde{\chi}) \mid \widetilde{\chi} \in \mathrm{Irr}(\widetilde{W}_{\mathbb{G}}) \text{ and } \mathrm{Res}_{W_{\mathbb{G}}}^{\widetilde{W}_{\mathbb{G}}}(\widetilde{\chi}) \in \mathrm{Irr}(W_{\mathbb{G}})\} \subseteq \mathrm{Class}(W_{\mathbb{G}}F\}.$$

Note that, in general, this set is not a basis of $\mathrm{Class}(W_{\mathbb{G}}F)$ as the elements are not linearly independent. With this we can define a map

$$\mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid -) : \mathrm{Class}(W_{\mathbb{G}}F) \to \mathrm{Class}(G,1) = \mathbb{C}\mathcal{E}(G,1)$$

by setting

$$\mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid f) = \frac{1}{|W_{\mathbb{G}}|}\sum_{w\in W_{\mathbb{G}}} f(wF)R_{\mathbb{T}_w}^{\mathbb{G}}(1).$$

To see why this construction has nice properties we need the following

Mackey Formula: If $\mathbb{L}, \mathbb{M} \leqslant \mathbb{G}$ are F-stable Levi subgroups then

$$^*R_{\mathbb{L}}^{\mathbb{G}} \circ R_{\mathbb{M}}^{\mathbb{G}} = \sum_g R_{\mathbb{L}\cap {}^g\mathbb{M}}^{\mathbb{L}} \circ^* R_{\mathbb{L}\cap {}^g\mathbb{M}}^{{}^g\mathbb{M}} \circ (\mathrm{ad}\, g).$$

Just as in the usual Mackey formula the sum is taken over double coset representatives of $L\backslash G/M$. However, one only considers double cosets which ensure that $\mathbb{L} \cap {}^g\mathbb{M}$ is the Levi complement of a parabolic subgroup of $\mathbb{G}$.

Theorem 3.1 (Deligne, Deligne–Lusztig, Bonnafé, Bonnafé–Michel, Taylor) *If $Z(\mathbb{G})$ is connected then the Mackey formula holds unless G has a composition factor $E_8(2)$.*

The Mackey formula has several important consequences:

- The map $\mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid -) : \mathrm{Class}(W_{\mathbb{G}}F) \to \mathrm{Class}(G,1)$ is an isometry onto its image.
- Just as in the case of tori we can define a map

  $$W_{\mathbb{G}}(\mathbb{L}) = N_{\mathbb{G}}(\mathbb{L})/\mathbb{L} \ni w \mapsto \mathbb{L}_w \leq \mathbb{G} \text{ an } F\text{-stable Levi subgroup.}$$

  Up to conjugacy we can assume $\mathbb{T}_0 \leq \mathbb{L}$ and $\mathbb{T}_0 \leq \mathbb{L}_w$. We then have the following identity relating Deligne–Lusztig induction and the map $\mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid -)$

  $$R_{\mathbb{L}_w}^{\mathbb{G}} \circ \mathcal{R}_{\mathbb{T}_0}^{\mathbb{L}_w}(1 \mid -) = \mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid -) \circ \mathrm{Ind}_{W_{\mathbb{L}}wF}^{W_{\mathbb{G}}F}.$$

Example: Consider the case where $\mathbb{G} = \mathrm{GL}_n$ is the general linear group then $W_{\mathbb{G}} = \mathfrak{S}_n$. If $f \in \mathrm{Irr}(W_{\mathbb{G}}F)$ is an irreducible character of the coset then $\pm\mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid f) \in \mathrm{Irr}(G)$ is irreducible. Moreover, in this case, we have $\mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid -)$ defines an isomorphism of $\mathrm{Class}(W_{\mathbb{G}}F)$ onto $\mathrm{Class}(G, 1)$.

Let us further assume that F acts as the identity on $W_{\mathbb{G}}$ so that $G = \mathrm{GL}_n(q)$ for some prime power q. In this case we have $\mathrm{Ind}_{W_{\mathbb{L}}wF}^{W_{\mathbb{G}}F} = \mathrm{Ind}_{W_{\mathbb{L}}w}^{W_{\mathbb{G}}}$ is simply induction from the coset $W_{\mathbb{L}}w \subseteq W_{\mathbb{G}}$. Coset induction has an adjoint given by coset restriction. If $W_{\mathbb{L}} = \mathfrak{S}_a$ and $w = (a+1, \dots, n)$ is an $(n-a)$-cycle then we have

$$\mathrm{Res}_{W_{\mathbb{L}}w}^{W_{\mathbb{G}}} = \mathrm{Res}_{\mathfrak{S}_a(a+1,\dots,n)}^{\mathfrak{S}_n}$$

is given by the Murnaghan–Nakayama formula.

Theorem 3.2 (Lusztig '85) *If $f \in \mathrm{Irr}(W_{\mathbb{G}}F)$ is irreducible then $\mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid f)$ is an explicit almost character (defined by the Fourier transform).*

Now, we say $f \in \mathrm{Class}(G)$ is **absolutely cuspidal** if ${}^*R_{\mathbb{L}}^{\mathbb{G}}(f) = 0$ for all $\mathbb{L} < \mathbb{G}$. We can generalise the construction above

$$\mathcal{R}_{\mathbb{T}_0}^{\mathbb{G}}(1 \mid -) \quad \rightsquigarrow \quad \mathcal{R}_{\mathbb{L}}^{\mathbb{G}}(f \mid -)$$

where $f \in \mathrm{Class}(L)$ is an absolutely cuspidal almost character.

Goal: Show that $\mathcal{R}_{\mathbb{L}}^{\mathbb{G}}(f \mid -)$ maps irreducible characters to almost characters.

To achieve this we follow Asai's approach: use Harish-Chandra information, Lusztig's result, integrality, and a lot of combinatorics.