Fourier matrices for unipotent characters

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Joint with Broué-Malle-Michel (1993) and Bonnafé-Rouquier.

We wish to understand Fourier matrices for finite reductive groups. Two interesting bases for class functions on $G(\mathbb{F}_q)$:

- (algebraic) characters of irreducible representations,
- (geometric) character functions of character sheaves.

The change of basis matrix is the Fourier matrix.

1 Drinfeld double of a finite group

Let Γ be an abstract finite group. Consider

$$\mathbb{C}^\Gamma = \{\text{functions } \Gamma \to \mathbb{C}\}$$

basis by Dirac δ_x, $x \in \Gamma$. This vector space has an action of Γ, so we can define

$$D_\Gamma := \mathbb{C}^\Gamma \rtimes \Gamma,$$

an algebra of dimension $|\Gamma|^2$.

Fact: There is a natural action of $SL_2(\mathbb{Z})$ on $Z(D_\Gamma)$ ($CF(D_\Gamma)$). If $x, y \in \Gamma$ with $xy = yx$,

$$[x : y] := \sum_{z \in \Gamma} {}^z\left(\delta_x \cdot y\right) = \sum_{z \in \Gamma} \left(\delta_{zxz^{-1}} \cdot zyz^{-1}\right).$$

Then $\{[x : y]\}_{x,y \in \Gamma}$ defines a basis of $Z(D_\Gamma)$.

The action of $SL_2(\mathbb{Z})$ on $Z(D_\Gamma)$ is induced by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] := [x^a y^b : x^c y^d].$$

Example

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} [x : y] = [x : xy] = \underbrace{\left(\sum_{r \in \Gamma} \delta_r \cdot r\right)}_{\text{Drinfeld element}} [x : y].$$

Another basis is given by Irr D_Γ:

$$D_\Gamma\text{-mod} \Leftrightarrow \bigoplus_{x \in \Gamma/\sim} C_\Gamma(x)\text{-mod},$$

$$X \rightsquigarrow X = \bigoplus_{x \in \Gamma} \delta_x X.$$

$$\operatorname{Irr} D_\Gamma \leftrightarrow \{[x : \chi]\}_{\substack{x \in \Gamma \\ \chi \in \operatorname{Irr} C_\Gamma(x)}}$$

Example

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} [x:\chi] = \left(\sum_{r \in \Gamma} \delta_r \cdot r \right) [x:\chi] = \omega_\chi(x)[x:\chi] = \frac{\chi(x)}{\chi(1)}[x:\chi].$$

Theorem 1.1 (Lusztig, Shoji) *There exists*

- *a partition*

$$\mathrm{Uch}(G(\mathbb{F}_q)) = \bigsqcup_{\mathcal{F} \text{ families}} \mathrm{Uch}(\mathcal{F}),$$

- *a finite group* $\Gamma_\mathcal{F}$ *attached to each family,*
- *a bijection*

$$\mathrm{Uch}(\mathcal{F}) \overset{1:1}{\leftrightarrow} \mathrm{Irr}\, D_{\Gamma_\mathcal{F}},$$

such that

(1)

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$$

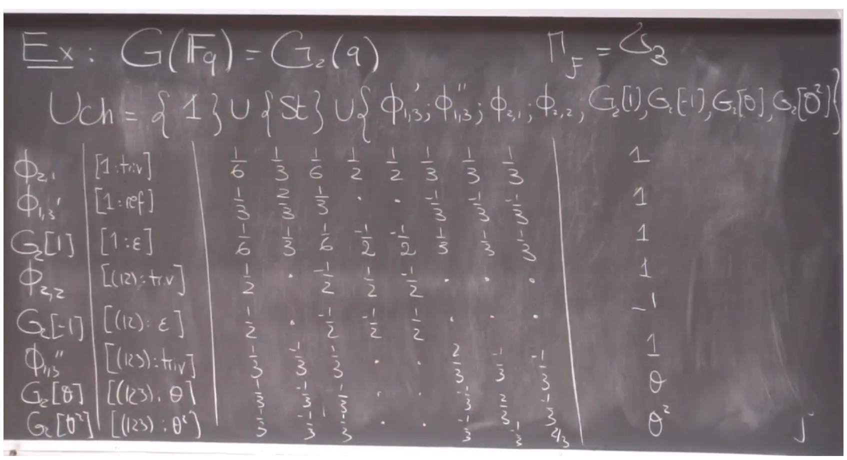
maps unipotent characters in $\mathcal{F}$ *to character functions of character sheaves of* $\mathcal{F}$*,*

(2)

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

acts on $\rho_{[x:\chi]}$ *by* $\omega_\chi(x)$ *which is the eigenvalue of the Frobenius on* $\rho_{[x:\chi]}$*.*

Example



2 Cohomology of Deligne-Lusztig

Let G be a connected reductive group/$\mathbb{F}_q$, with Frobenius $F : G \to G$, $G(\mathbb{F}_q) = G^F$, W the Weyl group, and assume that F acts trivially.

To $x \in W$ we associate $X(x)$ an algebraic variety with an action of $G(\mathbb{F}_q)$.

Then we have the associated cohomology $H^\bullet(X(x))$ which are $\overline{\mathbb{Q}}_\ell$ vector spaces with a linear action of $G(\mathbb{F}_q)$.

There is a map

$$H^i(X(x)) \xrightarrow{T_y} H^i(X(yxy^{-1}))$$

when we consider $x, y \in B_W$, the associated braid group to W (this map is not well-defined when we take $x, y \in W$), and an associated linear action of $C_{B_W}(x)$.

Example

$$X(1) = G(\mathbb{F}_q)/B(\mathbb{F}_q),$$

where $B(\mathbb{F}_q)$ is the Borel,

$$H^\bullet(X(1)) = \overline{\mathbb{Q}}_\ell[G(\mathbb{F}_q)/B(\mathbb{F}_q)],$$

which has an associated action of the Hecke algebra $\mathcal{H}_q(W)$ of W, (quadratic relation $(T_s - q)(T_s + 1) = 0$).

$$\begin{array}{ccc} \mathbb{Z}C_{B_W}(1) & \twoheadrightarrow & \mathcal{H}_q(W) \\ \| & & \\ \mathbb{Z}B_W & & \end{array}$$

T_x acts as F (Frobenius) on $H^\bullet(X(x))$.

$$\bigoplus_{x \in B_W} H^\bullet(X(x))$$

$$\sum_{y \in B_W} \delta_y T_y \text{ acts as the Frobenius.}$$

Recall that the Fourier transform converts the Unipotent characters into unipotent character sheaves.

$$\begin{array}{ccc} \text{Unipotent char} & \xleftrightarrow{\text{Fourier}} & \text{Unipotent char sheaves} \\ \uparrow & & \uparrow \\ \text{principal series} & & \text{almost characters} \\ \updownarrow & & \updownarrow \\ \operatorname{Irr} W & & \operatorname{Irr} W \end{array}$$

ρ_χ for $\chi \in \operatorname{Irr} W$ $\qquad$ $R_\chi, \chi \in \operatorname{Irr} W$

$\operatorname{Tr}(gT_x \mid H^\bullet(X(1))$ $\qquad$ $\operatorname{Tr}(gT_1 \mid H^\bullet(X(x))$

We have

$$\overline{\mathbb{Q}}_\ell[G(\mathbb{F}_q)/B(\mathbb{F}_q)] = \bigoplus_{\chi \in \operatorname{Irr} W} \rho_\chi \otimes \chi_q.$$

Consider the weight filtration of

$$H^\bullet(X(x))$$

$$H^i_c(X(x)) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} H^{i,j}_c(X(x)).$$

Here $C_{B_W}(x)$ acts on the LHS, and F acts with eigenvalues λq^j for $j \in \frac{1}{2}\mathbb{Z}$, $|\lambda| = 1$.

Define, for every $x, y \in B_W$ with $xy = yx$,

$$H(x:y)_{t,q} = g \mapsto \sum_{i,j} (-1)^i \operatorname{Tr}\left(gT_x \mid H^{i,j}_c(X(y))\right) t^j \tag{1}$$

for $g \in G(\mathbb{F}_q)$. We view this as an element of $\overline{\mathbb{Q}}_\ell[t^{\pm 1}, q^{\pm 1}]\,\mathrm{Uch}\,W$.

We use

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}$$

to define the action of this matrix:

Lusztig says the eigenvalues of F on unipotent characters is generic.

$$Fr : \mathrm{Uch}\,W \to \overline{\mathbb{Q}}_\ell\,\mathrm{Uch}\,W$$

$$\rho \mapsto \omega_\rho \rho.$$

Using equation1 we get

$$Fr\,(H(x,y))_{tq,t} = H(xy : y)_{t,q}.$$

$${}^{Sh} \cdot H(x : y)_{t,qt} = H(x : xy)_{t,q}.$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$Fo = Fr^{-1} Sh Fr^{-1}$$

$$Fo(H(x,y))_{q^{-1},t} = H(y^{-1}, x)_{t,q}.$$