The Loewy structure of certain fixed point algebras

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Let F be a field (algebraically closed of characteristic $p > 0$ if necessary). Let A be a finite-dimensional F-algebra, $\mathcal{J}(A)$ the Jacobson radical.

The **Loewy length** is

$$\mathrm{LL}(A) = \min\{t \in \mathbb{N}_0 \mid \mathcal{J}(A)^t = 0\}.$$

1 Background

Conjecture 1.1 *(Donovan ∼1975, open)*

For any finite p-group P, there are only finitely many Morita equivalence classes of blocks of finite groups with defect group isomorphic to P.

(There are infinitely *isomorphism* classes.)

Consequence (also open): There is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{LL}(B) \leq f(|P|)$ for every block B of a finite group with defect group $\simeq P$.

Question: Which f do we take? Note that $\mathrm{LL}(B) \leq |P|$ is NOT true in general.

Observation (Koshitani-Sambale-Kuelshammer 2014 & Sambale 2014): There is a function $g : \mathbb{N} \to \mathbb{N}$ such that $|P| \leq g(\mathrm{LL}(B))$ for every block B of a finite group with defect group P.

2 Centers of Blocks

Recall that knowing the center of a block gives a lot of information about the block itself, e.g., the number of irreducible Brauer characters.

Let B be a block of a finite group G with defect group D of order p^d.

Theorem 2.1 *(implicit in Passman 1980)*

$$\mathrm{LL}(Z(B)) \leq p^d + p^{d-1} + \cdots + p + 1.$$

Theorem 2.2 *(Okuyama 1981)*

$$\mathrm{LL}(Z(B)) \leq p^d = |D|.$$

Theorem 2.3 *(Otokita 2017)*

$$\mathrm{LL}(Z(B)) \leq |D| - \frac{|D|}{\exp(D)} + 1,$$

where $\exp(D)$ is the maximal order of an element of D.

Theorem 2.4 *(Otokita-Sambale-Kuelshammer 2018?)*

- $$\mathrm{LL}(Z(B)) = \frac{p^d - 1}{e} + 1 \text{ if } D \text{ is cyclic and } e \text{ is the inertial index (this is really a classical result),}$$

  $$\mathrm{LL}(Z(B)) \leq p^{d-1} + p - 1 \text{ if } D \text{ is noncyclic,}$$

  $$\mathrm{LL}(Z(B)) \leq \min\{p^{d-1}, 4p^{d-2}\} \text{ if } D \text{ is nonabelian.}$$

- $$\mathrm{LL}(Z(B)) \leq d^2 \exp(D) \text{ if } d \neq 0.$$

Theorem 2.5 *(Kuelshammer-Sambale 2018?)*
If B is a block with an abelian defect group D, then

$$\mathrm{LL}(Z(B)) \leq \mathrm{LL}(FD).$$

(This result verifies a consequence of Broué's conjecture.)

Corollary 2.6 *Let B be a block with an abelian defect group D, and suppose that the inertial quotient I of B acts semiregularly on $[D, I] \setminus \{1\}$. Then*

$$\mathrm{LL}(Z(B)) = \mathrm{LL}(Z(F[D \rtimes I])) = \mathrm{LL}((FD)^I),$$

where

$$(FD)^I = \{x \in FD \mid {}^h x = x \text{ for all } h \in I\}$$

is the fixed point algebra.

(The inertial quotient of B is a p'-subgroup of $\mathrm{Aut}(D)$.)

Remark There is a related result by Brough-Schwabrow.

3 Fixpoint algebras

(Breuer-Héthelyi-Horváth-Kuelshammer)

General Problem: Let P be a finite (p-)group acted on by a finite (p'-)group H. Determine the Loewy structure of the fixpoint algebra $(FP)^H$.

Special case: Henceforth we will fix a prime p, and $n, e \in \mathbb{N}$ such that $e \mid p^n - 1$.

Let

$$G := G(p, n, e) := \{\begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix} \mid \alpha, \beta \in \mathbb{F}_{p^n}, \alpha^e = 1\}.$$

Then $G = P \rtimes H$ where P is elementary abelian of order p^n, and H is cyclic with $|H| = e$.

Let $A := A(p, n, e) := (FP)^H$. Then $\dim A = 1 + z$ where $z = \dfrac{p^n - 1}{e}$.

Want: Loewy structure of $A = A(p, n, e)$.

Example

(i) For $n = 1$, $A \cong F[X]/(X^{z+1})$ is uniserial, with Loewy structure:

$$\begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix}$$

(ii) If $e = 1$, then $A = FP \cong F[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$.

(iii) For $p = 13, n = 2, e = 8$, we have $\dim A = 22$, $\mathrm{LL}(A) = 7$ and the Loewy structure is

$$\begin{matrix} 1 \\ 4 \\ 5 \\ 4 \\ 5 \\ 2 \\ 1 \end{matrix}$$

Proposition 3.1 • *A is commutative and symmetric (so the dimension of the socle of A is 1).*

- *A has an F-basis $b_0, b_1, \ldots, b_z$ such that*
$$b_k b_\ell \in \{0, b_{k+\ell}\}$$
for all k, ℓ.

- $$b_k b_\ell = b_{k+\ell} \iff i_t + j_t < p \text{ for } t = 1, \ldots, n$$
where
$$ke = \sum_{t=1}^{n} i_t p^{t-1} \text{ and } \ell e = \sum_{t=1}^{n} j_t p^{t-1}$$
are the p-adic expansions of ke and ℓe (i.e., there is no carry in adding ke and ℓe in the p-adic expansions).

Example Let $p = 11, n = 2, e = 15$.

$$2e = 30 = \underline{2} \cdot 11 + \underline{8} \cdot 1 \implies b_2 b_2 = 0 \quad \text{carry!}$$

$$3e = 45 = \underline{4} \cdot 11 + \underline{1} \cdot 1 \implies b_2 b_3 = b_5 \quad \text{no carry!}$$

Proposition 3.2 *Let $\lambda(0) := 0$,*

$$\lambda(k) := 1 + \max\{\lambda(\ell) : \ell \in \mathrm{L}_k\} \text{ for } k = 1, \ldots, z,$$

where L_k is the set of all $\ell \in \{0, 1, \ldots, k-1\}$ such that the digits in the p-adic expansions

$$ke = \sum_{t=1}^{n} i_t p^{t-1} \text{ and } \ell e = \sum_{t=1}^{n} j_t p^{t-1}$$

satisfy $j_t \leq i_t$ for $t = 1, \ldots, n$.

Then
$$b_k \in \mathcal{J}(A)^{\lambda(k)} \setminus \mathcal{J}(A)^{\lambda(k)+1}.$$

In particular,
$$\mathrm{LL}(A) = \lambda(z) + 1.$$

Isomorphisms: Let $n', e' \in \mathbb{N}$ such that $n' \mid n$ and $e = e' \frac{p^n - 1}{p^{n'} - 1}$. Then

$$A(p, n, e) \cong A(p, n', e').$$

Corollary 3.3
$$A \text{ is uniserial} \iff \frac{p^{n-1}}{p-1} \mid e$$

Question: What is $\mathrm{LL}(A(p, n, e))$?

Example (1) If $n = 1$, then $\mathrm{LL}(A) = \frac{p-1}{e} + 1$.

(2) If $n = 2$ and $p \neq 2$, then $\mathrm{LL}(A) \in \{\frac{p^2-1}{e_1} + 1, \frac{p^2-1}{2e_1} + 1\}$ where $e_1 = \gcd(e, p-1)$.
More precisely,

$$\mathrm{LL}(A) = \frac{p^2 - 1}{e_1} + 1 \iff e_1 \geq e_2 := \gcd(e, p+1) \text{ or both } e \text{ and } \frac{p^2-1}{e} \text{ are even.}$$

Proposition 3.4 *(i)*

$$\mathrm{LL}(A) \leq \left\lfloor n\frac{p-1}{m} \right\rfloor + 1$$

where

$$m = \min\left\{ s_p(ke) \mid k = 1, \ldots, z = \frac{p^n - 1}{e} \right\}$$

and $s_p(x)$ *is the* p*-adic digit sum of* $x \in \mathbb{N}_0$.

(ii) Equality holds in each of the following cases:

- $n \leq 3$,
- $e \leq 12$ *or* $e \geq \frac{p^n-1}{8}$,
- $e \mid \Phi_d(p)$ *for some* $d \in \{1, \ldots, 5\}$ *(where* Φ_d *denotes cyclotomic polynomials),*
- $m \leq 2$ *or* $m \mid p-1$ *or* $m = ne_1$,
- *many special examples.*

There are also examples where the inequality in Proposition 3.4 is strict.