Galois theory of periods, and the André-Oort conjecture

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Outline of Galois theory of periods

$\int_{\Delta} \omega$, Δ and ω "algebraic"

(ω diff. form on an algebraic variety X defined over some number field k, $\Delta \subset X(\mathbb{R})$ defined by algebraic inequations $/k$),

Transcendence of periods? Algebraic relations between them (period relations)?

Leibniz (1691, letters to Huygens): speculation about transcendence of π and some other (1-dim) periods. Inquiry about "accidental" cases when they are algebraic: "nothing happens without a reason"...

General conjectures:

Grothendieck (1966): *any period relation is of motivic origin.*

X smooth $/k$, $H_*(X(\mathbb{C}), \mathbb{Q}) \otimes H^*_{dR}(X) \to \mathbb{C}$

expressed by period matrix Ω_X.

If X proper, Z alg. subvariety dim. r of X^m ,

$\omega \in H^{2r}_{dR}(X^m) \subset H_{dR}(X)^{\otimes m} \rightsquigarrow \int_Z \omega \in (2\pi i)^r k$

conjecturally, period relations always come in this way.

Kontsevich (1998) (-Zagier): *any period relation comes from the basic rules for* $\int$:

linearity, product, algebraic change of variable $\int_\triangle f^*\omega = \int_{f_*\triangle} \omega$, Stokes $\int_\triangle d\omega = \int_{\partial\triangle} \omega$.

When made precise, these two conjectures can be proven to be equivalent.

Remark. Functional analog of periods: $\mathbb{Q} \rightsquigarrow \mathbb{C}(t)$. *Ayoub* (2015) *proved* analogs of Grothendieck's and Kontsevich's conjectures in this case.

Motives: categorification of the Grothendieck ring of varieties $K_0(Var_k)$.

$\rightsquigarrow$ abelian $\otimes$-category $MM(k)$.

(3 unconditional, compatible theories: A. (pure case), Nori, Ayoub; *cf.* Bourbaki nov. 2015).

eg. X smooth $\rightsquigarrow \langle X\rangle_\otimes \cong Rep_{\mathbb{Q}}\, G_X$

$G_X \subset GL(H(X(\mathbb{C})), \mathbb{Q})$ motivic Galois group of X.

$\langle X\rangle_\otimes \stackrel{H_B, H_{DR}}{\to} Vec_{\mathbb{Q}}$ $(k = \mathbb{Q}) \rightsquigarrow \Pi_X$ period torsor

Period pairing $\leftrightarrow$ canonical point in $\Pi_X(\mathbb{C})$:

$$\mathrm{Spec}\,\mathbb{C} \stackrel{\varpi_X}{\to} \Pi_X.$$

Grothendieck's period conjecture:

$\mathbf{PC}_X$: ϖ_X *is a generic point.*

Equivalently: Π_X *is connected, and*

$$\mathrm{TrDeg}_{\mathbb{Q}}\,\mathbb{Q}[\Omega_X] = \dim G_X.$$

If so, one can develop a bit of Galois theory of periods: $G_X(\mathbb{Q})$ would act on $\mathbb{Q}[\Omega_X] \rightsquigarrow$ Conjugates of periods...

Examples: $X = \mathbb{P}^1$: $G_X(\mathbb{Q}) = \mathbb{Q}^\times$,

$\mathbb{Q}[\Omega_X] = \mathbb{Q}[2\pi i]$ ($\mathbf{PC}_X$: Lindemann),

$X =$ CM elliptic curve by K: $G_X(\mathbb{Q}) = K^\times$,

$\mathbb{Q}[\Omega_X] = \mathbb{Q}[\omega_1, \eta_1]$ ($\mathbf{PC}_X$: Chudnovsky).

What if $k \subset \mathbb{C}$ is no longer algebraic over $\mathbb{Q}$?

generalized $\mathbf{PC}_X$: $\mathrm{TrDeg}_{\mathbb{Q}}\, k[\Omega_X] \geq \dim G_X$

(A. 1997). In the case of 1-motives $[\mathbb{Z}^n \to \mathbb{G}_m^n]$, this amounts to Schanuel's conjecture: if $x_1, \ldots, x_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent,

$$\mathrm{TrDeg}_{\mathbb{Q}}\, \mathbb{Q}[x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}] \geq n.$$

Outline of the **AO** conjecture.

Geometry of $\mathcal{A}_g$, the algebraic variety which parametrizes principally polarized abelian varieties of dimension g (e.g. $\mathcal{A}_1 = j$-line).

Special subvarieties of $\mathcal{A}_g$: subvarieties which parametrize PPAV with "extra symmetries".

PPAV with maximal symmetry (complex multiplication) are parametrized by *special points*.

AO conjecture: *special subvarieties of* $\mathcal{A}_g$ *are characterized by the density of their special points*.

Remarks. - $\mathcal{A}_g$ and its special subvarieties share a common geometric nature: they are *Shimura varieties*

$g = 1$:

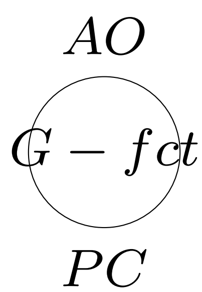
$$\begin{array}{ccc} \mathbb{C} & & \mathfrak{H} \\ \downarrow \wp & & \downarrow j \\ E \cong \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) & & \mathbb{A}^1 \cong \mathfrak{H}/SL_2(\mathbb{Z}) \end{array}$$

- "extra symmetries" ? ... Prescribed endomorphisms on A, or more generally, prescribed Hodge cycles on powers of A; looks transcendental, but is an algebraic condition: amounts to prescribe algebraic cycles on product of powers of A and some compact abelian pencils (A. 1996).

The AO conjecture is now a *theorem* (2015), after two decades of collaborative efforts putting together many different areas. Some key contributors: A. Yafaev, E. Ullmo, B. Klingler, J. Pila, J. Tsimerman...

Connections between **AO** and **PC**.

1. Early circle of ideas which gave rise to the AO conjecture.



G-functions $\leftrightarrow$ periods

Possible approach to **PC** (for abelian periods)?

Example. $E_\lambda : \ y^2 = x(x-1)(x-\lambda)$,

$$\omega_1(\lambda) \sim \pi F(\lambda), \eta_1 \sim \pi F'(\lambda), \quad F = F(\tfrac{1}{2}, \tfrac{1}{2}, 1; \lambda).$$

Diophantine theory of special values of G-functions F, F' $\rightsquigarrow$ new proof of **PC** for CM elliptic curves (A. (1996)).

For λ singular modulus (ie special point), $F(\lambda)(F'(\lambda)+\alpha F(\lambda)) = \beta/\pi,\ \alpha,\beta \in \bar{\mathbb{Q}}$. One cannot eliminate π... other solutions of the HGE are useless (log singularity at 0). But for AV of dim. $g > 1$ instead parametrized by a curve in $\mathcal{A}_g$ (instead of λ-line), one may get enough G-functions and relations between their special values.

Existence of lots of special points on the curve would allow to apply G-function theory efficiently. But analogy with Manin-Mumford renders the existence of ∞ly many special points unlikely in the non-modular case!

This was one source of my formulation of **AO** (1989) (Oort's later but independent formulation came from another source): **AO** bounds the hope for an application of G-fct. theory to **PC**; nevertheless, connections between **AO** and **PC** turn out to be more intricate.

2. Curves in products of modular curves

Case of $C \subset \mathbb{A}^1 \times \mathbb{A}^1 \subset \mathcal{A}_2$ (A. (1993) - first, and only, unconditional case of **AO**, until Pila (2011)):

AO$_{\mathbb{A}^1\times\mathbb{A}^1}$: *if C contains ∞ly many pairs of singular moduli (j, j'), C is either a vertical or horizontal line or some $X_0(N)$.*

i) (j_n, j'_n) singular moduli on C, (D_n, D'_n) (discriminants of quadratic orders). Class field theory $\rightsquigarrow$ for $n >> 0$, $\mathbb{Q}(\sqrt{D}_n) = \mathbb{Q}(\sqrt{D}'_n)$ and D'_n/D_n takes finitely many values.

ii) Linear forms in elliptic periods $\rightsquigarrow$ if ∞ly many special points on C, a branch of C goes to (∞, ∞): if $(j_n = j(\tau_n), j'_n) \rightarrow (\infty, j' = j(\tau'))$, then $\tau' = \omega'_1/\omega'_2$ is well-approximated by quadratic numbers τ_n; contradicts Masser's lower bound for $|\omega'_1 - \tau_n \omega'_2|$.]

iii) analysis of Puiseux expansions.

3. Hypergeometric values

$a, b, c \in \mathbb{Q}$, $\Re(c) > \Re(b) > 0$, $n = \text{den}(a, b, c)$,

$$F(a, b, c; \lambda) = \sum \frac{(a)_m (b)_m}{(c)_m \, m!} \lambda^m$$

$$= \frac{\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a} dx}{B(b, c-b)}$$

satisfies HG diff. equation, monodromy = Schwarz triangle group Δ.

numerator = period of $J^{new}_{n,a,b,c,\lambda}$

$$y^n = x^{n(b-1)}(1-x)^{n(c-b-1)}(1-\lambda x)^{-na}$$

denominator $B(b, c-b)$ = period of simple CM quotient $F_{b,c}$ of Fermat jacobian.

Question (J. Wolfart): *for which* (a, b, c) *are there* ∞*ly many* $\lambda \in \bar{\mathbb{Q}}$ *with* $F(a, b, c; \lambda) \in \bar{\mathbb{Q}}$*?*

Answer (Wüstholz-Wolfart-Cohen-Edixhoven - Yafaev): *iff* Δ *finite or arithmetic.*

[“if” due to Wolfart. “Only if”: 3 steps:

i) Wüstholz (special case of **PC**): $\bar{\mathbb{Q}}$-*linear* relations between periods of abelian periods come from endomorphisms

$\rightsquigarrow (\lambda, F(a, b, c; \lambda) \in \bar{\mathbb{Q}}) \Rightarrow J^{new}_{n,a,b,c,\lambda} \sim F_{b,c}$,

ii) for $\mathbb{P}^1 \setminus \{0, 1, \infty\} \xrightarrow{\phi} \mathcal{A}_g : \lambda \mapsto J^{new}_{n,a,b,c,\lambda}$,

$Im(\phi)$ special iff Δ finite or arithmetic.

iii) **AO** $\rightsquigarrow Im(\phi)$ special iff $J^{new}_{n,a,b,c,\lambda}$ has CM for ∞ly many λ's.]

4. Bialgebraicity

$\mathfrak{H}_g \quad \subset \mathfrak{H}_g^\vee$ (lagrangian grassmanian)

$\mathrm{j} \downarrow \qquad\qquad \tau = \Omega_1 \Omega_2^{-1} \mapsto \mathrm{j}(\tau)$

$\mathcal{A}_g$

Both $\mathfrak{H}_g^\vee$ and $\mathcal{A}_g$ are algebraic varieties/$\mathbb{Q}$, but j is transcendental.

Bialgebraic characterization of special subvarieties (Wüstholz-Cohen-Shiga-Wolfart-Ullmo-Yafaev): *$S \subset \mathcal{A}_g$ is special iff both S and a branch of $\mathrm{j}^{-1}(S) \subset \mathfrak{H}_g^\vee$ are algebraic and defined over $\bar{\mathbb{Q}}$.*

CSW: case of dim 0: $\tau, \mathrm{j}(\tau) \in \bar{\mathbb{Q}} \Leftrightarrow \mathrm{j}(\tau)$ is a special point.

Remark. $\mathfrak{H}_g^\vee$ and $\mathcal{A}_g$ are *transcendentally* related by $\mathfrak{j}$, but are also *algebraically* related by the relative period torsor:

$$\begin{array}{ccc} \Pi_g & \xrightarrow{\rho} & \mathfrak{H}_g^\vee \\ \downarrow & & \\ \mathcal{A}_g & & \end{array}$$

Π_g is a Sp_{2g}-torsor on $\mathcal{A}_g$, with coordinates corresponding to $\begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix}$, and ρ is the Sp_{2g}-equivariant surjective map $\begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix} \mapsto \Omega_1 \Omega_2^{-1}$.

5. Minimal special subvarieties

Given a PPAV A of dim. g, ie a point $\mathrm{j}_A \in \mathcal{A}_g$, there is a (unique) minimal special subvariety S_A containing j_A.

Question (Wolfart): *if A is defined over $\bar{\mathbb{Q}}$, what is the dimension of S_A?*

Answer: $\mathbf{PC}_A \Rightarrow \dim S_A = \mathrm{TrDeg}._{\mathbb{Q}}\,\mathbb{Q}(\tau)$, for any $\tau \in \mathfrak{H}$ such that $\mathrm{j}(\tau) = \mathrm{j}_A$).

via an analysis of (a reduction of) the relative period torsor Π_g. CSW is the case '0= 0" of this equality.

6. Other connections. One breakthrough in the proof of **AO** was the introduction of o-minimal methods (Pila-Zannier). In such counting arguments, an exceptional (semi-)algebraic set is left out. To handle it, one needs some functional transcendance results, which are functional analogs of the generalized **PC**.

Another breakthrough was to obtain lower bound for Galois orbits of special points from an average version of Colmez' conjecture on Faltings heights of CM abelian varieties. The connection (Tsimerman) uses a result of Masser-Wüstholz in the framework of transcendence of abelian logarithms.