Talk MSRI March 2017 Talk by S. Bloch

Motivic Γ-functions

Work of V. Golyshev and collaborators

M. Vlasenko, D. van Straten, S. Bloch, D. Zagier, ...

I. Recursion $\mathcal{D} = \mathbb{C}[t, t^{-1}, D]$ $D = t\frac{d}{dt}$

differential operators on $\mathbb{G}_m$.

$M = \mathcal{D}/\mathcal{D}\xi$ Holonomic $\mathcal{D}$-module, $\xi \in \mathcal{D}$.

$\xi = \sum_j t^j p_j(D)$ Assume $\exists$ $A(t) = \sum_{n \geq 0} A_n t^n$ single-valued

solution near $t = 0$.

$0 = \sum A_{m-j} P_j(m-j) t^m$ Recursion rel'n for $\{A_n\}$.

ex. $\xi = D - t$ $A(t) = e^t = \sum \frac{t^n}{n!}$; $nA_n - A_{n-1} = 0$

ex. (more interesting) $A_n = $ Const. term in $(x + a + \frac{1}{x})^n$

Recursion: $nA_n - a\cdot(2n-1)A_{n-1} + (a^2-4)(n-1)A_{n-2} = 0$.

II Periods associated to connections on curves (Bloch, H. Esnault – Homology for irregular connections on curves)

C smooth complete curve $/\mathbb{C}$ $\quad \phi \neq S \subset C$ finite subst.

M connection on $U := C - S$ (algebraic)

$\nabla : M \longrightarrow M \otimes \Omega^1_U \qquad H^1_{DR}(M, \nabla) := \mathrm{Coker}(\nabla) = \dfrac{M \otimes \Omega^1}{\nabla(M)}$

$\widetilde{\mathcal{O}}_U$ = analytic fncts on universal cover of U.

$\varepsilon : M \longrightarrow \widetilde{\mathcal{O}}_U \qquad \varepsilon(D(m)) = D(\varepsilon(m))$

$\varepsilon :=$ <u>Solution</u> of M

σ path on C $\quad \varepsilon|_\sigma$ assume single valued with rapid decay at pts of S and 0 bndry.

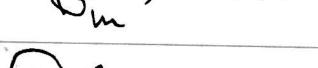
Then $\varepsilon|_\sigma$ represents class in $H^{rd}_1(M^\vee)$

<u>Thm</u> Perfect pairing $H^1_{DR}(M, \nabla) \times H^{rd}_1(M^\vee) \xrightarrow{\langle\,\rangle} \mathbb{C}$

$$\langle m \otimes \omega, \varepsilon|_\sigma \rangle = \int_\sigma \langle m, \varepsilon \rangle \omega$$

<u>ex.</u> $M = \mathcal{O}_{\mathbb{G}_m}, \quad \nabla(1) = -s\dfrac{dt}{t} + dt, \quad m = 1, \quad \omega = \dfrac{dt}{t}$

$M^\vee = \mathcal{O}_{\mathbb{G}_m}, \quad \varepsilon(1) = t^s e^{-t}$

$\sigma =$  to ∞

$$\text{period} = \int \langle m, \varepsilon|_\sigma \rangle \frac{dt}{t} = \int t^s e^{-t} \frac{dt}{t} = (e^{2\pi i s} - 1)\Gamma(s)$$

III Motivic Γ's

M, ∇_M on $U \subset \mathbb{G}_m$ Zariski open

$\nabla_s = \nabla_M + s\frac{dt}{t}$ Mellin transform

$m \in M$, ε sol'n for M, $\frac{dt}{t} \in \Omega^1_U$ (note $\varepsilon \cdot t^{-s}$ soln for ∇_s)

$$\text{Period} = \int_\sigma \langle m, \varepsilon \rangle t^{-s} \frac{dt}{t} =: \Gamma_{mot}(s) \text{ (depends on } M, m, \varepsilon|_\sigma)$$

$\xi = \sum t^j p_j(D)$ eqn. satified by m.

Then Γ_{mot} satisfies same recursion as sol'n of ξ

$$\sum p_j(s-j)\Gamma_{mot}(s-j) = 0$$

Compute $\frac{d}{ds}\Gamma_{mot}(s)\Big|_{s=0} = -\int_\sigma \langle m, \varepsilon \rangle \log t \frac{dt}{t}$

Log connection $\text{Log} = \mathcal{O}_{\mathbb{G}_m} e_0 \oplus \mathcal{O}_{\mathbb{G}_m} e_1$

$\nabla_{\text{Log}}(e_0) = e_1 dt$, $\nabla_{\text{Log}}(e_1) = \frac{-1}{t} e_1 dt$ $\quad \ell := \log(t) e_0^\vee + \frac{1}{t} e_1^\vee$ solution.

Then $\frac{d}{ds}\Gamma_{mot}(s)\Big|_{s=0} = -\int_\sigma \langle m \otimes e_0, \varepsilon \otimes \ell \rangle \frac{dt}{t}$ = $-$period of $M \otimes \text{Log}$.

IV Mahler Measure $\mathbb{G}_m^{n+1} \xrightarrow{pr_{n+1}} \mathbb{G}_m$

$(x_1, \dots, x_n, u) \longmapsto u$

$f(x_1, \dots, x_n)$ Laurent polynomial $\quad X: f-u=0 \subset \mathbb{G}_m^{n+1}$

$A := \Gamma(\mathbb{G}_m^{n+1} - X, \mathcal{O})$; $M = H^n_{DR}(\mathbb{G}_m^{n+1} - X / \mathbb{G}_m)$

$$= A \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} \Big/ d\left(\sum A \frac{dx_1}{x_1} \wedge \dots \wedge \widehat{\frac{dx_i}{x_i}} \wedge \dots \wedge \frac{dx_n}{x_n}\right)$$

$m = (1 - f/u) \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} \in M$, $\varepsilon = \int_C$

$C = \{|x_1| = \dots = |x_n| = 1\}$ family of n-chains on $\mathbb{G}_m^{n+1} / \mathbb{G}_m^n$

(avoid $u \in \mathbb{G}_m$ s.t. $C \cap \{f=u\} \neq \phi$)

View C as family of paths on $\mathbb{G}_m^{n+1} - X / \mathbb{G}_m$.

σ = clsd path on $\mathbb{P}^1 - f(C) - \{0, \infty\} - S$

(S = finite set of u s.t. $f(x_1, \dots, x_n) = u$ degenerates)

σ winding no $+1$ around $f(C)$, 0 around $0, \infty$.

Mahler measure $m(f) := \frac{1}{(2\pi i)^n} \int_C \log|f| \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}$

Thm. $2\pi \cdot m(f) = \mathrm{Im}\left(\frac{d}{ds} \Gamma_{mot}(s) \Big|_{s=0}\right)$.

V Apéry Program (Apéry, Beukers, Van ~~Straten~~ Straten, M. Kerr, ...)

Use recursion associated to homogeneous and inhomogeneous sol'ns of Picard Fuchs eqns. to approximate limits of Beilinson regulator values.

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MSRI

Spencer Bloch

Motivic Gamma Functions
+ Recursion

Originated with V. Golyšev + collaborators

Loeser & Sabbah – finite difference eqns

I: Recursion:

$$\mathcal{D} = \mathbb{C}[t, t^{-1}, D] \qquad D = t\frac{d}{dt}$$

Holonomic $\mathcal{D}$-modules $\mathcal{D}/\mathcal{D}\cdot\xi$

$$\xi = \sum_{\text{fte}} t^j P_j(D)$$

$$\begin{array}{ccc} \mathcal{X}_0 & \subseteq & \mathcal{X} \\ \text{smooth} \downarrow & & \downarrow f \\ U & \subseteq & \mathbb{P}^1 \end{array}$$

Gauss-Manin connection on $H^n_{DR}(\mathcal{X}/U)$ ... a $\mathcal{D}$-module

Solution: $M = \mathcal{D}/\mathcal{D}\xi$

$$s : M \longrightarrow \tilde{\mathcal{O}}_U$$

"algebra of multivalued functions on U"

$\mathbb{C}[[T]]$ some ps algebra

$D = t\frac{d}{dt}$ acts

$$s(Dm) = D(s(m))$$

$$s(m) = \sum_{n\geq 0} A_n T^n$$

$m := 1 \in M = \mathcal{D}/\xi\mathcal{D}$

$\xi \cdot 1 = \xi = 0$ in M.

So $s(m) = \sum_{n\geq 0} A_n t^n$ satisfies

diff'l equation $\xi(\sum_{n\geq 0} A_n t^n)$.

Recursion satisfied by $\{A_n\}$. By $(*)$

$$\sum_{m,j} A_{m-j} P_j (m-j)\, t^m = 0.$$

<u>Ex:</u> If $A_n(a) :=$ constant term in $(x + a + 1/x)^n$

Then $\sum_n A_n(a) t^n$ satisfies a

DE $\xi(t)$. Recursion:

$$n A_n(a) - a(2n-1) A_{n-1}(a) + (a^2-4)(n-1) A_{n-2}(a) = 0$$

II Periods associated to connections on curves

Bloch - Esnault: Irregular connections on curves ...

C smooth complete curve / $k \subseteq \mathbb{C}$.

$U = C - S$; S finite

$\nabla : M \to M \otimes \Omega^1_U$ connection

$\rightsquigarrow H^1_{DR}(M, \nabla)$

$:= \operatorname{coker}\{\Gamma(U, M) \to \Gamma(U, M \otimes \Omega^1)\}$

$m \otimes \omega \in M \otimes \Omega^1_U$ ... think of as in $H^1_{DR}(M, \nabla)$

$m \in M$, $\omega \in \Omega^1_U$

<u>Periods</u>: elets are lin combs of s/σ rapidly decaying (Deligne, Malgrange)

$$H^1_{DR}(M, \nabla) \otimes H_1^{rd}(\check{M}) \longrightarrow \mathbb{C}$$

algebraic object

dual connection

σ path in C, $m \otimes \omega \in H^1_{DR}(M)$

s soln $\in \check{M} \otimes \tilde{\mathcal{O}}$ analytic object

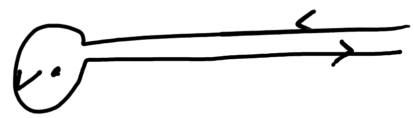
s/σ single-valued + rapid decay at cusps S.

$$\langle m \otimes \omega, s/\sigma \rangle = \int_\sigma \langle m, s \rangle \cdot \omega$$

<u>Ex</u>: $M = \mathcal{O}_U$, $\nabla(1) := -s \frac{dt}{t} + dt$

$s \in \mathbb{C}$

$s = \text{sol}^{\underline{n}}$ $\quad s(1) = t^s e^{-t}$

σ : 

So

$$\int \langle m, s\rangle\, \omega = \int t^s e^{-t} \frac{dt}{t} = \left(e^{2\pi i s} - 1\right)\Gamma(s)$$

rapid decay at ∞

$\rightsquigarrow$ Motivic gamma functions ... but need regular singularities (which don't have here).

III Motivic Γ-functions

M, ∇ on $U = \mathbb{P}^1 - S$, $\quad 0, \infty \in S$.

$t \in \mathcal{O}(U)$, $m \in M$ $\quad$ (s now ε)

Periods: $\displaystyle\int_\sigma \langle \varepsilon, m\rangle\, \omega$

Mellin Transform: $\quad M t^s$

$$\nabla(m t^s) = (\nabla m) t^s + s\, m \frac{dt}{t}$$

i.e. $M \rightsquigarrow (M, \nabla) \otimes \left(1 \mapsto s \frac{dt}{t}\right)$

$$\Gamma_{(M, mot; \varepsilon/_\gamma)}(s) := \int_\sigma \langle m, \sigma \rangle t^{-s} \frac{dt}{t}$$

Thm: If solution $\sum A_n t^n$ satisfies recursion $\sum P_j(n-j) A_{n-j} = 0$, then $\Gamma_{mot}(s)$ satisfies

$$\sum P_j(s-j) \Gamma_{mot}(s-j) = 0$$

NB: Some ambiguity: choice of ε, path σ

Why one cares about them:

$$\frac{d}{ds} \Gamma_{mot}(s) \Big|_{s=0}$$

Log: $\mathcal{O} e_0 \oplus \mathcal{O} e_1$ on $\mathbb{G}_m$

$$\nabla_{Log}(e_0) = e_1 \, dt, \quad \nabla_{log}(e_1) = \frac{-1}{t} e_1 \, dt$$

Solution: $\quad t \check{e}_0 + \frac{1}{t} \check{e}_1$

$M \otimes Log$: $\int \langle m, \varepsilon \rangle \log t \frac{dt}{t}$

$\uparrow$

period of $M \otimes Log$

Mähler measure:

$$\mathbb{G}_m^{n+1} \longrightarrow \mathbb{G}_m$$

$$(x_1, \dots, x_n, u) \longmapsto u$$

$f(x_1, \dots, x_n)$ Laurent poly

$X : f - u = 0$ in $\mathbb{G}_m$.

$$\mathbb{G}_m^{n+1} - X \xrightarrow{p_{n+1}} \mathbb{G}_m$$

$$M = H^n_{DR}((\mathbb{G}_m - X)/\mathbb{G}_m$$

$$= \frac{A \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}}{d\left(\sum A \frac{dx_1}{x_1} \wedge \dots \wedge \frac{\widehat{dx_j}}{x_j} \wedge \dots \wedge \frac{dx_n}{x_n}\right)} \qquad A := \Gamma(\mathbb{G}_m^{n+1} - X, \mathcal{O})$$

$m \in M = H_{DR}$

$$m = (1 - f/u) \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} \in M$$

$$\varepsilon = \int_C \qquad \int_\sigma \langle \varepsilon, m \rangle \log t \frac{dt}{t}$$

$$C = \{ |x_1| = \dots = |x_n| = 1 \}$$

Assume $C \cap X = \phi$

finite

σ closed path in $\mathbb{P}^1 - (\{0, \infty\} \cup S \cup f(C))$

winds 1 time around $f(C)$, 0 times around $0, \infty$

Mähler Measure

$$:= \frac{1}{(2\pi i)^n} \int_C \log |f| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$

$$= \frac{-1}{2\pi} \operatorname{Im} \left(\frac{d}{ds} \Gamma_{mot}(s) \Big|_{s=0} \right)$$

IV Applications to Apéry program.

(Irrationality of $\zeta(2), \zeta(3)$ ..

Recurrences associated to inhomogenous solutions of your PF connection.