1. Elliptic multiple zeta values and periods by Nils Matthes

Multiple zeta values are real numbers defined by

$$\zeta(k_1, \cdots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$$

where $k_i \geq 1$ and $k_1 \geq 2$.

The multiple polylogarithms are multivalued functions on $\mathbb{P}^1$ minus three points, and their value at 1 are equal to multiple zeta values. The multiple zeta values occur as coefficients of Drinfeld associators. The periods of mixed Tate motives over $\mathbb{Z}$ are multiple zeta values.

Goal. To generalize this picture to genus 1.

Motto. Multiple zeta values and elliptic multiple zeta values arise from parallel transport.

Example 1.1. The basic example is the number $2\pi i$. Consider $\mathbb{C}^\times$. Its fundamental group $\pi_1(\mathbb{C}^\times; 1)$ is isomorphic to the free abelian group $\mathbb{Z} \cdot \gamma$ where γ denotes the unit circle counterclockwise. Consider the trivial vector bundle $\mathbb{C} \times \mathbb{C}^\times \to \mathbb{C}^\times$ on $\mathbb{C}^\times$ with the connection ∇ given by

$$\nabla f := df - \frac{dz}{z}$$

where f is a section of the bundle and z is the coordinate on $\mathbb{C}^\times$. The map

$$\pi_1(\mathbb{C}^\times; 1) \to \mathbb{C}, \quad \gamma \mapsto \int_\gamma \frac{dz}{z} = 2\pi i \cdot 1$$

induces an isomorphism

$$\pi_1(\mathbb{C}^\times; 1) \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{C}.$$

It shows that $2\pi i$ comes from parallel transport on $\mathbb{C}^\times$ along the connection ∇.

$$\pi_1(\mathbb{C}^\times; 1) \simeq \mathbb{Z}(1)$$

and $2\pi i$ is a period of $\mathbb{Z}(1)$.

Now we discuss multiple zeta values (MZVs).

Example 1.2. Take $X := \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$. Then its fundamental group $\pi_1(X; x)$ is equal to the free group $\mathrm{Fr}(\gamma_0, \gamma_1)$ generated by the loops γ_0, γ_1 around 0 and 1 respectively. Let $\mathbb{C}\langle\langle x_0, x_1 \rangle\rangle$ denote the ring of power series in the non-commutative variables x_0, x_1 where the multiplication is by concatenation of variables. Consider the trivial bundle

$$\mathbb{C}\langle\langle x_0, x_1 \rangle\rangle \times X \to X$$

with the Knizhnik–Zamolodchikov connection

$$\nabla_{\mathrm{KZ}} f = df - \omega_{\mathrm{KZ}} f$$

where

$$\omega_{\mathrm{KZ}} = \frac{dz}{z} x_0 + \frac{dz}{z-1} x_1.$$

We get multiple zeta values via parallel transport along this connection.

Consider the map

$$T_x^{\mathrm{KZ}} : \mathbb{Q}[\pi_1(X;x)] \to \mathbb{C}\langle\langle x_0, x_1\rangle\rangle, \quad \gamma \mapsto \sum_{k=0}^{\infty} \int_\gamma \underbrace{\omega_{\mathrm{KZ}} \cdots \omega_{\mathrm{KZ}}}_{k\text{-times}}$$

where

$$\int_\gamma \omega_1 \cdots \omega_n := \int_{1 \geq t_1 \geq \cdots t_n \geq 0} \gamma^*(\omega_1)(t_1) \cdots \gamma^*(\omega_n)(t_n)$$

The map T_x^{KZ} highly depends on x as the values of the iterated integrals vary a lot with the choice of the base point x (unlike the $\mathbb{C}^\times$ case as in the previous example, where we would get $2\pi i$ if we choose a different base point).

We have to use Deligne's tangential base points (which means a tangent vector at one of the punctures $0, 1, \infty$). Take

$$\vec{1}_0 = \frac{\partial}{\partial z} \in T_0\mathbb{P}^1_{\mathbb{C}}, \quad -\vec{1}_1 = -\frac{\partial}{\partial z} \in T_1\mathbb{P}^1_{\mathbb{C}}.$$

The formalism of tangential base points gives a way of integrating from one tangent vector to another (which involves some regularization procedure).

We get a morphism

$$T^{\mathrm{KZ}}_{-\vec{1}_1, \vec{1}_0} : \mathbb{Q}[\pi_1(X; -\vec{1}_1, \vec{1}_0)] \to \mathbb{C}\langle\langle x_0, x_1\rangle\rangle, \quad \gamma \mapsto \sum_{k=0}^{\infty} \mathrm{Reg} \int_\gamma \underbrace{\omega_{\mathrm{KZ}} \cdots \omega_{\mathrm{KZ}}}_{k\text{-times}}.$$

Let $\mathrm{dch} \in \pi_1(X; -\vec{1}_1, \vec{1}_0)$ denote the canonical path from 0 to 1. Evaluating the map $T^{\mathrm{KZ}}_{-\vec{1}_1, \vec{1}_0}$ at dch we get the Drinfeld associator $\Phi(x_0, x_1)$.

$$T^{\mathrm{KZ}}_{-\vec{1}_1, \vec{1}_0}(\mathrm{dch}) = \Phi(x_0, x_1)$$

The coefficient of $x_0^{k_1-1}x_1 \cdots x_0^{k_n-1}x_1$ (for $k_1 \geq 2$) in $\Phi(x_0, x_1)$ is $(-1)^n\zeta(k_1, \cdots, k_n)$.

Example 1.3 (Elliptic multiple zeta values (eMZVs))**.** Consider the punctured elliptic curve

$$E_\tau^\times := (\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau) \setminus \{0\}$$

where τ is an element of the upper half-plane $\mathfrak{h}$. The fundamental group $\pi_1(E_\tau^\times; \rho)$ of $E_\tau^\times$ with respect to any base point ρ is equal to the free group $\mathrm{Fr}(\alpha, \beta)$ where α, β are as indicated in the talk. Let $\xi := r\tau + s$ denote a coordinate on $E_\tau^\times$ where $(r, s) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$.

Consider the trivial bundle

$$\mathbb{C}\langle\langle a, b\rangle\rangle \times E_\tau^\times \to E_\tau^\times$$

with the "elliptic Knizhnik–Zamolodchikov–Bernard connection"

$$\nabla_{\mathrm{KZB}} f = df - \omega_{\mathrm{KZB}} f$$

where

$$\omega_{\mathrm{KZB}} := dr \cdot a + 2\pi i \mathrm{ad}(a) e^{r \cdot \mathrm{ad}(a)} F_\tau(2\pi i \xi, \mathrm{ad}(a))(b) d\xi$$

with

$$F_\tau(\xi, \eta) := \frac{\Theta'_\tau(0)\Theta_\tau(\xi + \eta)}{\Theta_\tau(\xi)\Theta_\tau(\eta)}.$$

(Calaque–Enriquez–Etingof, Levin–Racinet, Hain–Matsumoto, Brown–Levin)

Need Deligne's theory of tangential base point to get something arithmetically nice.

We look at parallel transport with respect to the base point

$$\vec{v}_0 := (-2\pi i)^{-1} \frac{\partial}{\partial \xi} \in T_0(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau)$$

which corresponds to the tangent vector

$$-\frac{\partial}{\partial z} \in T_1(\mathbb{C}^\times / q^{\mathbb{Z}})$$

to the Tate curve $\mathbb{C}^\times / q^{\mathbb{Z}}$ at 1 where $q = e^{2\pi i \tau}, z = e^{2\pi i \xi}$.

Similar to as before, we get

$$T^{\mathrm{KZB}}_{-\vec{v}_0, \vec{v}_0} : \mathbb{Q}[\pi_1(E_\tau^\times; -\vec{v}_0, \vec{v}_0)] \to \mathbb{C}\langle\langle a, b \rangle\rangle, \quad \gamma \mapsto \sum_{k=0}^{\infty} \mathrm{Reg} \int_\gamma \omega^k_{\mathrm{KZB}}.$$

Define

$$A(\tau) := T^{\mathrm{KZB}}_{-\vec{v}_0, \vec{v}_0}(\alpha) \in \mathbb{C}\langle\langle a, b \rangle\rangle$$

$$B(\tau) := T^{\mathrm{KZB}}_{-\vec{v}_0, \vec{v}_0}(\beta) \in \mathbb{C}\langle\langle a, b \rangle\rangle$$

which are Enriquez elliptic KZB associators.

Definition 1.4 (Elliptic multiple zeta values)**.** *The $\mathbb{Q}$-vector spaces of A-elliptic and B-elliptic multiple zeta values are defined by*

$$\mathcal{EZ}^A := \mathrm{span}_{\mathbb{Q}} \{ A(\tau)|_w | w \in < a, b > \},$$

$$\mathcal{EZ}^B := \mathrm{span}_{\mathbb{Q}} \{ B(\tau)|_w | w \in < a, b > \}.$$

These are $\mathbb{Q}$-subspaces of the ring of holomorphic functions on the upper-half plane.

Question 1.5. What are eMZVs after all?

Focus on B-eMZVs.

Let $G_{2k}(\tau)$ denote the normalized Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2k$. Define

$$\mathcal{G}(2k_1, \cdots, 2k_n; \tau) := \int_\tau^{\vec{1}_\infty} G_{2k_1}(z_1) dz_1 \cdots G_{2k_n}(z_n) dz_n$$

where $k_i \geq 0$ and $G_0 := -1$. Since the Eisenstein is nonzero at ∞, we need some regularization (to make the integral converge), which was done by Brown.

Fact: Considered as a function of τ, the iterated integrals of Eisenstein series are linearly independent, *i.e.*, for any subfield K of $\mathbb{C}$, there is an isomorphism

$$\mathrm{span}_K \{ \mathcal{G}(2k_1, \cdots, 2k_n; \tau) \} \simeq K(g_0, g_2, g_4, \cdots)$$

which sends

$$\mathcal{G}(2k_1, \cdots, 2k_n; \tau) \mapsto g_{2k_1} \cdots g_{2k_n}.$$

Here $K(g_0, g_2, g_4, \cdots)$ denotes the free shuffle algebra.

Theorem 1.6 (Enriquez)**.** *Every B-eMZVs can be written as a $\mathcal{Z}$-linear combination of $\mathcal{G}(2k_1, \cdots, 2k_n; \tau)$ where $\mathcal{Z}$ denotes the $\mathbb{Q}$-algebra of multiple zeta values.*

This rewriting is unique.

We get an embedding

$$\psi^B : \mathcal{EZ}^B \hookrightarrow \mathcal{Z}\langle g_0, g_2, \cdots \rangle.$$

The main goal is to describe the image of ψ^B. To describe $\mathrm{Im}(\psi^B)$, we need the following definition.

Definition-Proposition 1.7 (Tsunogai)**.** *For every $k \geq 0$, there exists a unique derivation*

$$\varepsilon_{2k} : \mathbb{L}(a, b) \to \mathbb{L}(a, b)$$

such that

(i) $\varepsilon_{2k}(a) = \mathrm{ad}^{2k}(a)(b)$,

(ii) $\varepsilon_{2k}([a, b]) = 0$,

(iii) $\varepsilon_{2k}(b)$ *has no linear term in* a.

Let $\mathfrak{u} := \mathrm{Lie}(\varepsilon_{2k})$ denote the Lie algebra generated by ε_{2k} inside $\mathrm{Der}(\mathbb{L}(a, b))$.

Conjecture 1.8. *As $\mathbb{Q}$-algebras,*

$$\mathcal{EZ}^B \simeq \mathcal{U}(\mathfrak{u})^\vee \otimes \mathcal{Z} \hookrightarrow \mathcal{Z}\langle g_0, g_2, \cdots \rangle.$$

Theorem 1.9 (Lochak–Matthes–Schneps)**.** *This conjecture is true "modulo $2\pi i$".*