Exponential motives

joint work with Peter Jossen (ETH)

inspired by Kontsevich, Nori, Bloch-Esnault...

X smooth variety $/k \hookrightarrow \mathbb{C}$

$f : X \to \mathbb{A}^1_k$ a regular function

- de Rham cohomology

$$\mathcal{E}^f = (\mathcal{O}_X, \nabla(1) = -df) \quad \text{flat rk 1 connection}$$

$$DR(\mathcal{E}^f) : \mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \cdots$$

$$\nabla(\omega) = d\omega - df \wedge \omega$$

Definition: $H^*_{dR}(X, f) = \mathbb{H}^*(X, DR(\mathcal{E}^f))$

→ The case where f is constant gives back usual de Rham cohomology of X.

→ The associated local system is $\mathbb{C} \cdot \exp(f)$ which is trivial (!) (Irregular singularities)

Usually, to define periods one looks at the analytification morphism and uses the Poincaré lemma. This does not work here, so we need to introduce rapid decay.

- Rapid decay cohomology

(classical in the study of asymptotics of differential equations, letter Deligne-Malgrange '76)

$$\begin{array}{c} X(\mathbb{C}) \\ \Big\downarrow f \\ \mathbb{C} \end{array}$$

$S_t = \{ \mathrm{Re}\, z \geq t \}$

$$H_*^{rd}(X, f) = \lim_{t \to \infty} H_*^{rd}(X(\mathbb{C}), f^{-1}(S_t); \mathbb{Q})$$

with respect to the inclusions $f^{-1}(S_{t'}) \subseteq f^{-1}(S_t)$ for $t \leq t'$.

$$\boxed{\gamma = (\gamma_t)_{t \in \mathbb{R}} \qquad \partial \gamma_t \subseteq f^{-1}(S_t)}$$

Again, f = constant gives back the usual singular cohomology of X.

But we can also think of γ as an unbounded cycle!

- Comparison isomorphism

$$H^n_{dR}(X,f) \otimes H^{rd}_n(X,f) \longrightarrow \mathbb{C}$$

$$[w] \otimes \gamma \longmapsto \lim_{t\to\infty} \int_{\gamma_t} e^{-f} w$$

(Deligne, Dimca-Saito, Sabbah, Hien-Roucairol)

exponential periods when $K \subseteq \overline{\mathbb{Q}}$

i) converges because f is very positive on the boundary of γ_t

ii) independent of representatives:

$$\left[\lim_{t\to\infty} \int_{\gamma_t} e^{-f}(d\eta - df \wedge \eta) = \lim_{t\to\infty} \int_{\gamma_t} d(e^{-f}\eta) \right.$$

$$\left. \underset{\text{Stokes}}{=} \lim_{t\to\infty} \int_{\partial\gamma_t} e^{-f}\eta = 0 \right]$$

Exponentials of algebraic numbers $X = \mathrm{Spec}(\mathbb{Q})$, $f = \alpha$

Examples

① $X = \mathbb{A}^1 = \mathrm{Spec}\, \mathbb{Q}[x]$ $\qquad f = a_n x^n + \dots + a_0 \in \mathbb{Q}[x]$

$(n \geq 2,\ a_n > 0)$

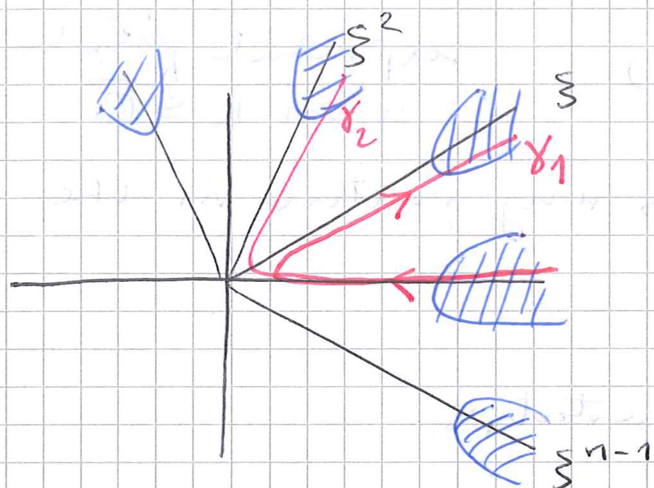
de Rham complex:

$$\mathbb{Q}[x] \xrightarrow{\nabla} \mathbb{Q}[x]\,dx$$

$$g \longmapsto (g' - f'g)\,dx$$

$$H^0_{dR}(X,f) = \operatorname{Ker}\nabla = 0$$

$$H^1_{dR}(X,f) = \operatorname{coker}\nabla = \langle dx, x\,dx, \dots x^{n-2}dx \rangle_{\mathbb{Q}}$$



$$H_1^{rd}(X,f) = \langle \gamma_1, \dots, \gamma_{n-1} \rangle_{\mathbb{Q}}$$

$$\xi = e^{\frac{2\pi i}{n}}$$

The matrix of the comparison isomorphism

$$\left(\int_{\gamma_i} x^{j-1} e^{-f(x)}\,dx \right)_{i,j=1,\dots n-1}$$

e.g. $f = x^2$ $\qquad \displaystyle\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$

$f = x^n$ $\qquad \displaystyle\int_{\gamma_i} x^{j-1} e^{-x^n}dx = \frac{\xi^{ij}-1}{n}\,\Gamma\!\left(\frac{j}{n}\right)$

$$\left(n\,\Gamma\!\left(\frac{j}{n}\right) = \int_{-\gamma_1-\gamma_2-\dots-\gamma_{n-1}} x^{j-1} e^{-x^n}dx \right)$$

<u>not</u> expected to be periods!

But periods can be constructed from tensor operations

e.g. $\Gamma(\frac{1}{n})^n = \prod_{i=1}^{n-1} B(\frac{1}{n}, \frac{i}{n})$ is a period of a

Fermat hypersurface.

<u>Conjecture</u> (Lang) If $n \geq 3$,

$$\mathrm{tr\,deg}\ \overline{\mathbb{Q}}\left(\Gamma(\tfrac{1}{n}), \ldots \Gamma(\tfrac{n-1}{n})\right) = 1 + \frac{\varphi(n)}{2}$$

André: follows from Grothendieck's period conjecture, but in a rather twisted way (!)

(2) Euler-Mascheroni constant

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right)$$

$$= - \int_0^\infty \log(x)\, e^{-x}\, dx$$

$$= \int_0^1 \int_0^1 e^{-xy}\, dx\, dy - \int_1^\infty \int_1^\infty e^{-xy}\, dx\, dy$$

suggests looking at $X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{Q}[x, y]$

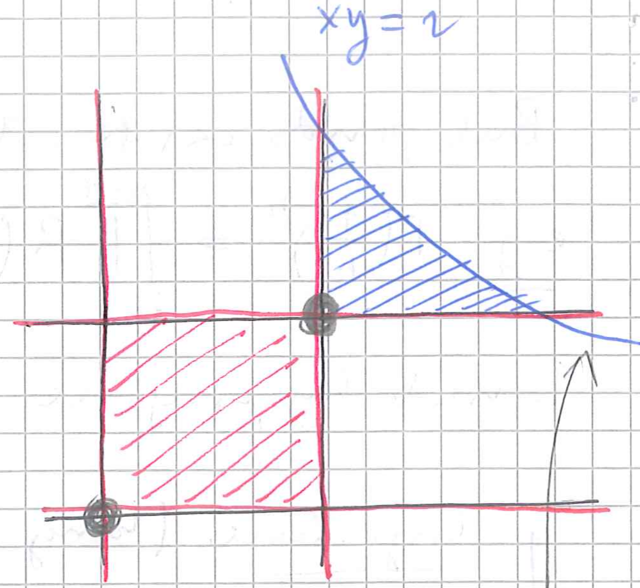
$Y = \{ xy(x-1)(y-1) = 0 \}$

$$f = xy$$

$$H_2(X, Y, f)$$

$$\int_{\square - \triangle} e^{-xy}\, dx\, dy = \gamma$$

a class in de Rham cohomology



there is a loop inside here

$$t \mapsto (e^{2\pi i t}, 2e^{-2\pi i t})$$

This motive has dimension 3.

$$\pi : \tilde{X} \longrightarrow X, \quad \tilde{Y} = \text{strict transform of } Y$$

$$\tilde{X} = Bl_{(1,1)} X \qquad \tilde{f} = f \circ \pi$$

$$\rightsquigarrow \pi^* : H^2(X, Y, f) \longrightarrow H^2(\tilde{X}, \tilde{Y}, \tilde{f})$$

The image of π^* is two-dimensional and fits into an extension

$$0 \to \mathbb{Q}(0) \to M \to \mathbb{Q}(-1) \longrightarrow 0$$

with period matrix $\begin{pmatrix} 1 & \gamma \\ 0 & 2\pi i \end{pmatrix}$.

Suggestive interpretation: Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \quad \text{near } s=1,$$

so γ is the regularized value $\zeta(1)$!

Question: K a number field

$$\zeta_K(s) = \frac{a}{s-1} + b + O(s-1) \qquad a \neq 0$$

define $\gamma_K = \frac{b}{a}$ (Ihara). Is γ_K an exponential period? Construct the corresponding extension (!)

Exponential motives

Theorem: There exists a $\mathbb{Q}$-linear neutral Tannakian category $M^{exp}(K)$ which is universal with respect to "rapid decay cohomology".
It contains usual Nori motives $M(K)$ as a full subcategory but the image is not stable under extension.

Objects have a weight filtration

pure objects $\leftrightarrow$ smooth varieties with a proper function!

e.g. $H^1(\mathbb{A}^1, f)$ is pure of weight one.

Sketch of the construction:

(1)

$$
\begin{tikzcd}
 & M^{exp}(k) \arrow[dr, "\text{forget}"] & \\
p: Q^{exp}(k) \arrow[ur, "\tilde{p}"] \arrow[rr] & & \mathrm{Vect}_{\mathbb{Q}}
\end{tikzcd}
$$

quiver representation

quiver with objects

$[X, Y, f, n, i]$

+ morphisms given by obvious functoriality and triples

- $[X', Y', f', n, i] \to [X, Y, f, n, i]$
- $[Y, Z, f|_Y, n-1, i] \to [X, Y, f, n, i]$

$$\mathrm{End}(p) = \lim_{\substack{Q \subseteq Q^{exp}(k) \\ \text{finite}}} \mathrm{End}(p|_Q)$$

$$\left\{ (e_q) \in \prod_{q \in Q} \mathrm{End}(p(q)) \;\middle|\; \begin{array}{ccc} p(q) & \longrightarrow & p(q') \\ {\scriptstyle e_q}\downarrow & & \downarrow{\scriptstyle e_{q'}} \\ p(q) & \longrightarrow & p(q') \end{array} \right\}$$

$M^{exp}(k)$ = category of fin. dim $\mathbb{Q}$-vector spaces together with a continuous $\mathrm{End}(p)$-action

$$M = (V, Q, \alpha : \mathrm{End}(p|_Q) \to \mathrm{End}(V))$$

Now: $M^{exp}(k)$ is an abelian category, universal for representations comparable with p.

② To define a $\otimes$-product, one looks at the Künneth formula

$$\bigoplus_{i+j=n} H^i(X_1, f_1) \otimes H^j(X_2, f_2) \longrightarrow H^n(X_1 \times X_2, f_1 \boxplus f_2)$$

Thom-Sebastiani sum

Basic lemma: X affine of dimension d, $Y \subsetneq X$ a closed subvariety, $f : X \to \mathbb{A}^1$ a function. There exists a closed subvariety $Z \subseteq X$ of dim $\leq d-1$ and containing Y such that

$$H^n(X, Y, f) = 0 \quad \text{for } n \neq d$$

$\rightsquigarrow$ allows one to define a $\otimes$-product

$$\therefore \quad M^{exp}(k) = \mathrm{Rep}(G^{exp}(k))$$

motivic exponential Galois group

$H^1(\mathbb{A}^1, x^2)^{\otimes 2} = H^1(x^2 + y^2 = 1)$

Given an object M of $M^{exp}(k)$, let $\langle M \rangle_{\otimes}$ be the smallest Tannakian subcategory containing it. Then $\boxed{\langle M\rangle_\otimes = \mathrm{Rep}(G_M)}$

$$G^{exp}(k) \twoheadrightarrow G_M$$

$\cap$

$GL(R(M))$

<u>Conjecture</u>: $\operatorname{tr\,deg}_{\overline{\mathbb{Q}}} \begin{pmatrix}\text{exponential} \\ \text{periods of} \\ M\end{pmatrix} = \dim_{\mathbb{Q}} G_M$

→ <u>Example</u> 1: Lindemann-Weierstrass theorem

$$\alpha_1, \dots \alpha_n \in \overline{\mathbb{Q}}^\times$$

<u>Thm</u>: $\operatorname{tr\,deg}_{\overline{\mathbb{Q}}}(e^{\alpha_1}, \dots e^{\alpha_n}) = \mathrm{rk}\,\underbrace{\langle \alpha_1, \dots \alpha_n\rangle}_{\substack{\text{subgroup of} \\ \overline{\mathbb{Q}} \text{ generated by} \\ \text{them}}}$

$$M = \bigoplus_{i=1}^{n} H^0(\mathrm{Spec}\, k, -\alpha_i)$$

$k = \mathbb{Q}(\alpha_1, \dots \alpha_n)$

You prove that the Galois group of M is a split torus of rank $\mathrm{rk}\langle \alpha_1, \dots \alpha_n\rangle$

→ <u>Example</u> 2: $H^1(\mathbb{A}^1, x^n) = M_n$

$$(M_n^{\otimes m})^{\mu_n} \simeq H^{m-2}_{prim}(\{x_1^n + \dots + x_m^n = 0\})(-1)$$

(Anderson) $\qquad$ <u>Fermat hypersurface</u>

$$\dim G_{M_n} = 1 + \frac{\varphi(n)}{2}$$

in accordance with the Lang conjecture.

2017.03.28 Javier Fresnan

Galois Theory of Exponential Periods

Joint $\bar{c}$ Peter Jossen (ETH)

Inspiration:
Kontsevich, Nori, Bloch-Esnault

<u>Setting</u>:

X smooth variety $/K$ $\quad K \hookrightarrow \mathbb{C}$

$f : X \to \mathbb{A}^1_K$ regular function

de Rham cohomology:

$$\mathcal{E}^f = (\mathcal{O}_X, \nabla(1) = -df)$$

$DR(\mathcal{E}^f)$:

$$\mathcal{O}_f \xrightarrow{d_f} \Omega^1_X \xrightarrow{d_f} \Omega^2_X \xrightarrow{d_f} \cdots$$

$$d_f(\omega) = d\omega - df \wedge \omega$$

<u>Def</u>: $H^i_{dR}(X, f) = H^i(X, DR(\mathcal{E}^f))$

fin dim K-vect-space.

Remarks:

① f constant: $H^\bullet_{dR}(X, f) = H^\bullet_{dR}(X)$

② The local system of horizontal sections is $\mathbb{C}\cdot \exp f$ (Irreg. sings)

Rapid decay cohomology (Deligne, Malgrange)

$$\begin{array}{l} X(\mathbb{C}) \\ f \big\downarrow \\ \mathbb{C} \end{array} \qquad t \in \mathbb{R} \qquad S_t = \{ \mathrm{Re}(z) \geq t \}$$

$$H_\bullet^{rd}(X, f) := \lim_{t\to\infty} H_\bullet(X(\mathbb{C}), f^{-1}(S_t); \mathbb{Q})$$

$$H^\bullet_{rd}(X, f) := \mathrm{Hom}(H_\bullet^{rd}(X, f), \mathbb{Q})$$

Comparison Isomorphism (Deligne, Sabbah, Hien-Roucairol)

$$H^n_{dR}(X, f) \otimes H_n^{rd}(X, f) \to \mathbb{C}$$

non-degenerate pairing.

X affine: ω alg diff form

$$[\omega] \otimes (\gamma_t)_{t\in\mathbb{R}_+} \mapsto \lim_{t\to\infty} \int_{\gamma_t} e^{-f}\omega$$

Values are exponential periods.
(Here $K \subseteq \overline{\mathbb{Q}}$.)

<u>Remark</u>: If $f = \text{const} \neq 0$, then the comparison isom is twisted by e^{-f}.

<u>eg</u>: $\alpha \in \overline{\mathbb{Q}}$, $K = \mathbb{Q}(\alpha)$, $X = \operatorname{Spec} K$

$f = e^{-\alpha} \rightsquigarrow e^{\alpha}$ is an exponential period.

<u>Examples</u>:

① $X = \mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[x]$.

$f = a_n x^n + \cdots + a_0$ $\quad n \geq 2$, $a_n > 0$

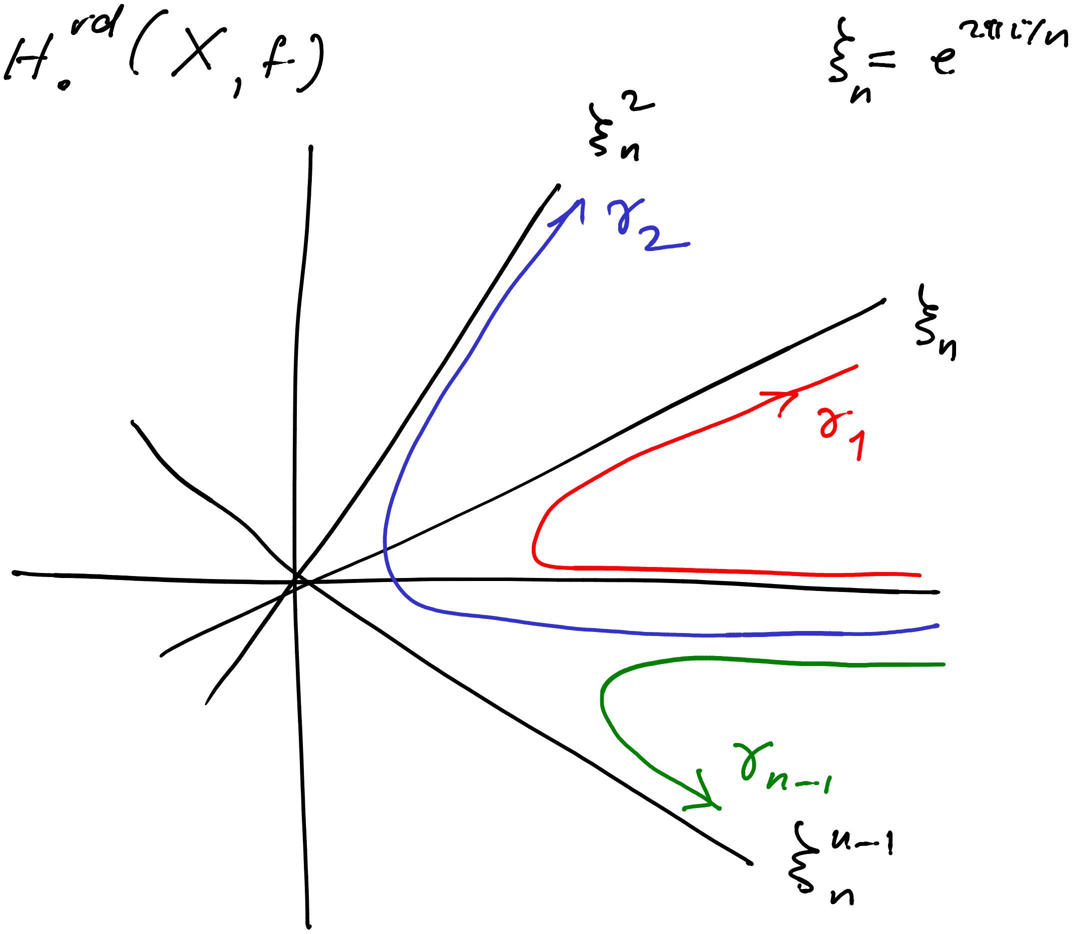
DR complex:

$$\mathbb{Q}[x] \xrightarrow{d_f} \mathbb{Q}[x]\,dx$$
$$g \longmapsto (g' - f'g)\,dx$$

$H^0_{dR}(X, f) = \ker d_f = 0$ $\quad \left\{ \begin{array}{l} \frac{g'}{g} = f' \\ g = ce^f \end{array} \right.$

$H^1_{dR}(X, f) = \langle dx, x\,dx, \ldots, x^{n-2}dx \rangle_{\mathbb{Q}}$

$H_\bullet^{rd}(X, f)$



$$H_1^{rd}(\mathbb{A}^1, f) = \langle \gamma_1, \dots, \gamma_{n-1} \rangle_{\mathbb{Q}}$$

Period matrix:

$$\left(\int_{\gamma_i} x^{j-1} e^{-f(x)}\, dx \right)_{i,j=1,\dots,n-1}$$

eg: $f = x^2$

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

- $f = x^n$

$$\int_{\gamma_1} x^{j-1} e^{-x^n} dx = \frac{\zeta_n^{ij} - 1}{n} \Gamma\left(\frac{j}{n}\right)$$

- $(\sqrt{\pi})^2 = \pi$ $\qquad x^2 + y^2 = 1$

- $$\Gamma\left(\frac{j}{n}\right)^n = (j-1)! \prod_{i=1}^{n-1} B\left(\frac{j}{n}, \frac{ij}{n}\right)$$

period of a Fermat hypersurface.

$$\Gamma\left(\frac{j}{n}\right) = \int_{-\gamma_1 - \cdots - \gamma_n} x^{j-1} e^{-x^n} dx$$

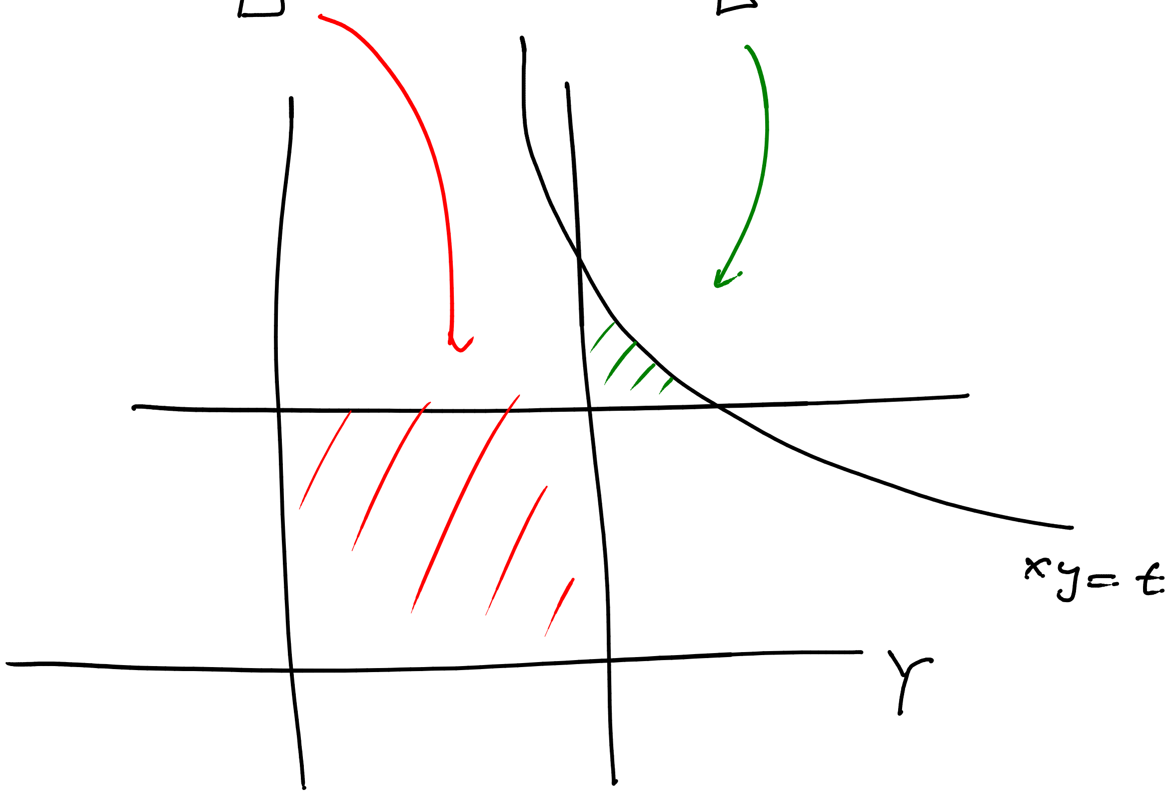
② Euler – Mascheroni constant

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k} - \log(n) \right)$$

$$= -\int_0^\infty \log(x) e^{-x} dx$$

$$= \int_0^1 \int_0^1 e^{-xy} dx dy - \int_1^\infty \int_1^\infty e^{-xy} dx dy$$

$$= \int_{\square} e^{-xy}\, dx\, dy - \int_{\Delta} e^{-xy}\, dx\, dy$$



$H_2^{rd}(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}(S_t))$ 3 dimensional

$\tilde{X} = Bl_{(1,1)} X \longrightarrow X$ $\quad s \mapsto (e^{2\pi i s}, t e^{-2\pi i s})$

$\tilde{Y}$ = strict transform

$\tilde{f} = f \circ \pi$

$\pi^* : H^2(X, Y, f) \longrightarrow H^2(\tilde{X}, \tilde{Y}, \tilde{f})$

$M := \operatorname{im} \pi^*$. It fits into exact seqce.

$$0 \to \mathbb{Q}(0) \to M \to \mathbb{Q}(-1) \to 0$$

$\begin{pmatrix} 1 & \gamma \\ 0 & 2\pi i \end{pmatrix}$ period matrix.

Suggestive interpretation.

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

$$\zeta(1)^{reg} = \gamma$$

<u>Question</u>: K # field

$$\zeta_K(s) = \frac{a}{s-1} + b + O(s-1) \quad a \neq 0$$

Ihara: $\gamma_K := \frac{b}{a}$. Is it an exponential period. Can we const an extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ with period γ over K?

<u>Exponential Motives</u>

<u>Thm</u>: ① There is a $\mathbb{Q}$-linear tannakian category $M^{exp}(K)$ which is universal w.r.t. rapid decay homology.

② It contains Nori motives $M(K)$ as a full subcategory, but is not closed under extensions

So $M^{exp}(K) = \mathrm{Rep}\ G^{exp}(K)$

exponential motivic Galois gp

M object $M^{exp}(K)$

$\langle M \rangle_{\otimes} = \mathrm{Rep}(G_M) \qquad G_M \subseteq GL\, H_{2d}(M)$

Conjecture:

$\mathrm{trdeg}\ \bar{\mathbb{Q}}\,(\text{periods } M) = \dim_{\mathbb{Q}} G_M$

Ex: Lindermann - Weierstrass Thm

$\alpha_1, \dots, \alpha_n \in \bar{\mathbb{Q}}^{\times}$

$\mathrm{trdeg}\ \bar{\mathbb{Q}}\langle e^{\alpha_1}, \dots, e^{\alpha_n} \rangle$

$= \dim_{\mathbb{Q}} \langle \alpha_1, \dots, \alpha_n \rangle$

$K = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$

$M = \bigoplus_{i=1}^{n} H^0(\mathrm{Spec}\, K, -\alpha_i \cdot)$

G_M = split torus of dim = dim $\langle \alpha_1, \dots, \alpha_n \rangle$.

② $M_n = H^1(\mathbb{A}^1, x^n)$

$$\operatorname{tr\,deg} \bar{\mathbb{Q}}\left(\Gamma\left(\tfrac{1}{n}\right), \dots, \Gamma\left(\tfrac{n-1}{n}\right)\right)$$

$$\overset{\text{Lang}}{\underset{??}{=}} \quad 1 + \varphi(n)/2 \qquad n \geq 3.$$

André: follows from Grothendieck's period conjecture.

PROP: (Anderson)

$$\left(M_n^{\otimes m}\right)^{\mu_n} \cong H^{m-2}_{prim}\left(\{x_1^n + \dots + x_m^n = 0\}\right)(-1)$$

$$\dim G_{M_n} = 1 + \varphi(n)/2.$$