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Geoffroy HOREL – Galois actions on operads

Motivated by Grothendieck's "Esquisse d'un programme"

Goal : understand $\Gamma = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Study Γ by its action on $\pi_1^{ét}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$.

Example: $X = \mathbb{G}_m$, we capture Γ^{ab}.

$X = \mathbb{G}_m - \{1\} = \mathbb{P}^1 - \{0, 1, \infty\} = \mathcal{M}_{0,4}$

The action of Γ on $\pi_1^{ét}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$ is faithful.

$$\pi_1^{ét}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}) = \widehat{F}_2$$

(Consequence of Belyi's theorem)

Take $\{\mathcal{M}_{g,n}\}_{g,n}$ + various structures relating these schemes/stacks.

"Teichmüller tower".

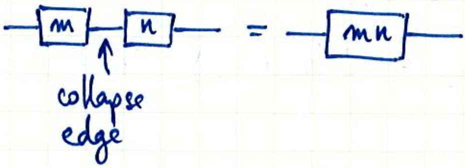
My goal : revisit Grothendieck's program.

1- Instead of using groups, use homotopy types.

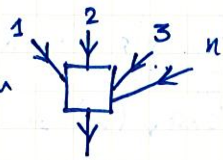
2- Use the language of operads in order to formalize glueing/erasing of points.

Operads

Monoid : M ; $m \in M$: 

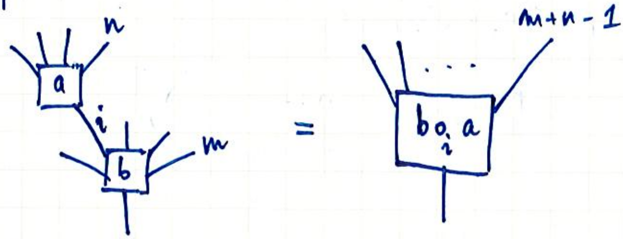


(associative)

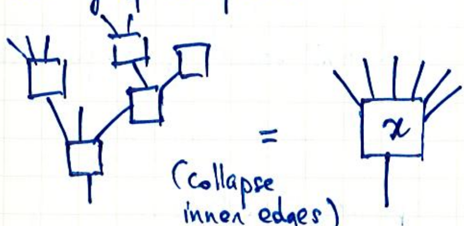
In an operad, I have operations of the form 

("arity n")

Composition :



Associativity of composition



(collapse inner edges)

(x does not depend on the order of collapsing)

<u>Def</u>: An operad $\mathcal{P}$ is a collection of sets (spaces) $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ equipped with

(1) an identity $1 \in \mathcal{P}(1)$

(2) $\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \longrightarrow \mathcal{P}(m+n-1) \quad \forall i \in \{1,..,n\}$

(3) $\Sigma_n \circlearrowright \mathcal{P}(n)$

satisfying conditions.

<u>Example 1</u>: X a set (space)

$\mathrm{End}_X(n) = \mathrm{Hom}(X^n, X)$.

$\mathrm{id} \in \mathrm{End}_X(1)$.

$$\left.\begin{array}{l} f \in \mathrm{Hom}(X^n, X) \\ g \in \mathrm{Hom}(X^m, X) \end{array}\right\} \begin{array}{l} f \circ_i g(x_1, \dots, x_{m+n-1}) \\ = f(-,-,\dots,\underbrace{g(-,\dots,-)}_{i},\dots,-) \end{array}$$

<u>Example 2</u>: $\mathcal{M}(n) = \mathcal{M}_0^{n+1}$ = moduli space of Riemann surfaces of genus 0 with $n+1$ boundary components.

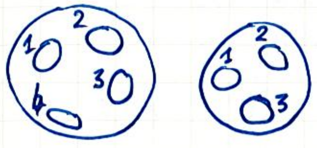
$\mathcal{M}_0^{n+1} \simeq B\Gamma_0^{n+1}$

$\{\mathcal{M}_0^{n+1}\} = \{\mathcal{M}(n)\}$ has an operad structure.

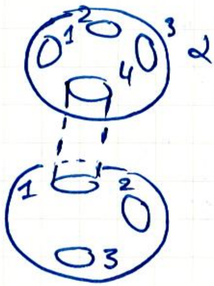
[the boundary components are parametrized by S^1]

Composition:

$\underset{\|}{\alpha} \in \mathcal{M}(\underset{\overset{\|}{3}}{n}), \ \underset{\|}{\beta} \in \mathcal{M}(\underset{\overset{\|}{2}}{m})$



$\beta \circ_1 \alpha \in \mathcal{M}(\underset{\overset{\|}{4}}{m+n-1}) \quad =$



<u>Example 3</u>: $\overline{\mathcal{M}}(n) = \overline{\mathcal{M}}_{0,n+1}$

$\{\overline{\mathcal{M}}(n)\}_{n\in\mathbb{N}}$ has an operad structure.

Grothendieck's program, revisited: study Γ via its action on the operads $\widehat{\mathcal{M}}$ and $\widehat{\overline{\mathcal{M}}}$ in the homotopy category of operads in profinite spaces.

<u>Profinite homotopy theory</u> (Artin-Mazur-Friedlander-...)

<u>Def</u>: A profinite homotopy type is a pro-object in the

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category of π-finite spaces.

π-finite space = topological space X such that $\pi_0(X) < \infty$ and $\forall x \in X, \bigoplus_i \pi_i(X,x) < \infty$.

Example: G a profinite group, $G = \lim_{i \in I} G_i$ ← finite group

$$B_{pro} G = \text{"}\lim_{i\in I}\text{"}\, BG_i .$$

Given $X = \text{"}\lim_{i\in I}\text{"}\, X_i$, $Y = \text{"}\lim_{j \in J}\text{"}\, Y_j$,

$$\mathbb{R}\mathrm{Map}(X,Y) = \operatorname{holim}_{j\in J} \operatorname{hocolim}_{i\in I} \mathbb{R}\mathrm{Map}(X_i, Y_j)$$

Def: The homotopy category $\mathrm{Ho}(\widehat{Sp})$ of profinite spaces.

Objects: profinite homotopy types

$\mathrm{Hom}(X,Y) = \pi_0 \mathbb{R}\mathrm{Map}(X,Y)$.

Adjunction:

$$\mathrm{Ho}(Sp) \rightleftarrows \mathrm{Ho}(\widehat{Sp}) : \operatorname{holim}$$
$$X \longmapsto \hat{X}$$

Proposition: There is a similar adjunction

$$\mathrm{Ho}(\mathrm{Op}(\mathcal{S}p)) \rightleftarrows \mathrm{Ho}(\mathrm{Op}(\widehat{\mathcal{S}p}))$$

<u>Theorem</u>: [P. Boavida de Brito, H., M. Robertson]

There is an injection

$$\Gamma \longrightarrow \mathrm{Aut}_{\mathrm{Ho}(\mathrm{Op}(\widehat{\mathcal{S}p}))}(\widehat{\mathcal{M}})$$

$$\parallel$$

$$\widehat{GT} = \text{profinite Grothendieck-Teichmüller group.}$$

[Ihara, Drinfeld]

<u>Theorem</u>: [BHR]

There is an action of $\widehat{GT}$ on $\widehat{\widehat{\mathcal{M}}}$ which is non trivial. It is such that the operad map $\widehat{\mathcal{M}} \to \widehat{\widehat{\mathcal{M}}}$ is $\widehat{GT}$-equivariant.