2017.03.29 Anton Alexeev

Introduction: $g=0$

k = field of char 0

	3D-top (braid gps)		# theory		Lie theory (BCH series)
torsor:	Assoc(k)		Formal MZVs		Sol KV
$\circlearrowleft$ group:	$GRT_1(k)$	$\xhookrightarrow{\text{Furusho}}$	DS	$\xhookrightarrow{\text{Schneps}}$	KRV
Lie algebra:	$\mathfrak{grt}_1(k)$	$\hookrightarrow$	$\mathfrak{ds}$	$\hookrightarrow$	$\mathfrak{krv}$
			double shuffle		

Enriquez–Torossian–Alexeev

Standard Conjecture: all inclusions are isomorphisms and

$$\mathfrak{grt}_1 \cong \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \dots)$$

$\mathfrak{krv} \hookrightarrow$ Special derivations D of $\mathbb{L}(x,y)$:

$$D(x) = [x,a], \quad D(y) = [y,b]$$

$$D(x+y) = [x,a] + [y,b] = 0$$

$$\mathrm{krv} = \left\{ D \in \mathrm{Der}\, \mathbb{L}(x,y) : D \text{ special and } \mathrm{div}(D) = \mathrm{tr}[f(x+y) - f(x) - f(y)] \text{ for some } f \in k[x] \right\}$$

to be explained

A = free assoc alg on x, y.

$|A| = A/[A,A]$ = cyclic words in x, y

$\mathrm{tr}: A \to |A|$ projection

Eg: $\mathrm{tr}(x^2y) = \mathrm{tr}(xyx) = \mathrm{tr}(yx^2)$

- $f(x) = \frac{1}{3}x^2$

$$\mathrm{tr}(f(x+y) - f(x) - f(y))$$
$$= \mathrm{tr}(x^2y + y^2x)$$

$$\alpha \in A \Rightarrow \alpha = \underset{\substack{\uparrow \\ k}}{\alpha_0} + (\partial_x \alpha)x + (\partial_y \alpha)y$$

à la Magnus

$$\mathrm{div}(D) := \mathrm{tr}\left(\partial_x D(x) + \partial_y D(y)\right)$$

Remark:

- original def^n due to Kashiwara – Vergne
- This version: F. Naef

<u>Thm</u>: $\operatorname{div}: \operatorname{Der}(\mathbb{L}(x,y)) \to |A|$

is a 1-cocycle:

$$\operatorname{div}([D_1, D_2]) = D_1 \cdot \operatorname{div}(D_2) - D_2 \operatorname{div}(D_1)$$

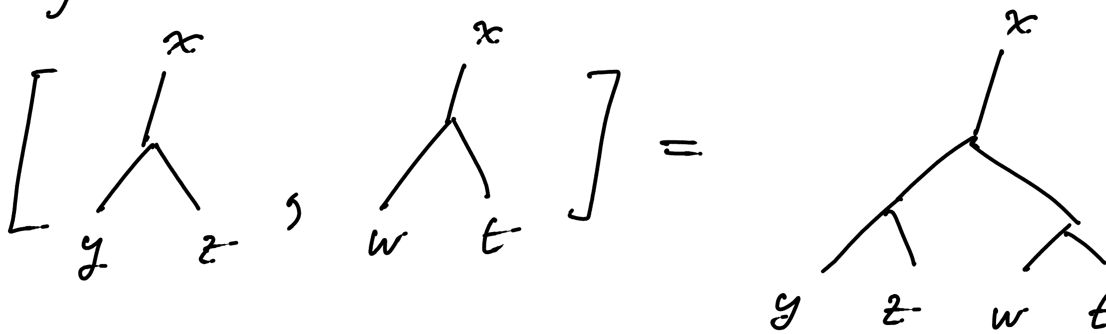
Exercises:

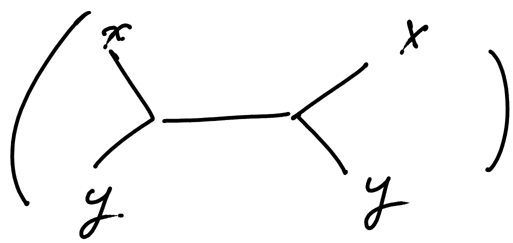
0: Check for divergence of vector fields

1: Show that $\mathfrak{krv}$ is a Lie sub alg of the special derivations.

<u>Graphical</u> <u>calculus</u>:

Lie bracket: only have 2 letters x, y. Introduce more ...



div (

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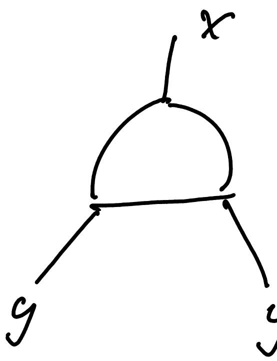
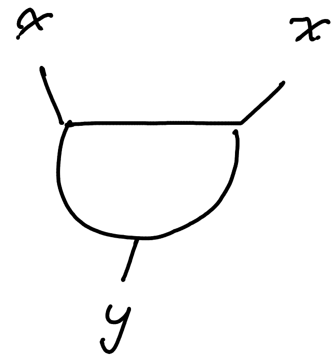
= 

+ 

$= \mathrm{tr}(x^2 y + y^2 x)$

<u>Rk:</u> This is σ_3 in special derivations

Q: What is the higher genus $g>0$ analogue of the Kashiwara – Vergne story?

<u>Higher genus story</u>

Joint with N. Kawazumi, Y. Kuno, F. Naef.

torsor	$\mathrm{Sol}\,KV_{g,n+1}$
group	$KRV_{g,n+1}$
Lie alg	$krv_{g,n+1}$

$\Sigma = \Sigma_{g,n+1}$ oriented 2-mfld with $n+1$ bdry cpts

$$krv_{0,3} = krv$$

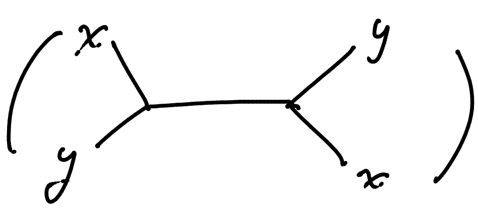
$$krv_{1,1} = \left\{ \begin{array}{l} D \in Der\ \mathbb{L}(x,y) : D[x,y] = 0 \\ div(D) = tr\ f([x,y]) \end{array} \right\}$$

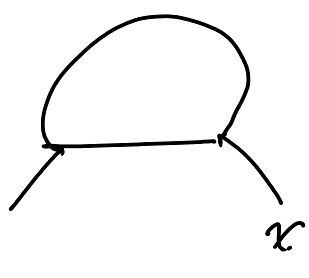
<u>Thm</u>: (AKKN)

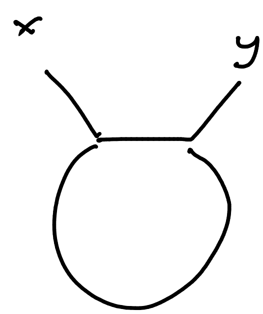
- $krv = krv_{0,3} \hookrightarrow krv_{1,1}$ (after completion)
- $\varepsilon_{2n} \in krv_{1,1}$ where

$$\varepsilon_{2n}(x) = ad_x^{2n}(y), \quad \varepsilon_{2n}([x,y]) = 0.$$

<u>Ex</u>:

div ()

= 

− 

$$= tr\,(yx - xy) = 0$$

(agrees with Enomoto-Satoh trace)

Goldman–Turaev Lie Bialgebras

$\Sigma = \Sigma_{g,n+1}$ oriented

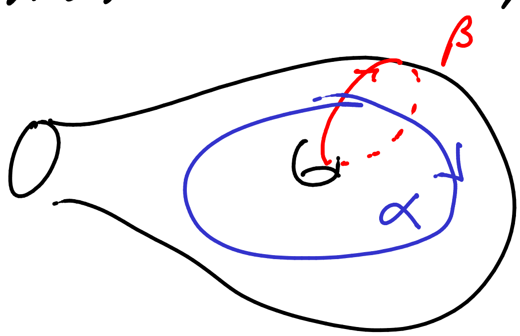
$\pi = \pi_1(\Sigma_{g,n+1})$, $k\pi$ group alg

$\mathcal{A} = k\pi / [k\pi, k\pi]$

$= H_0$ (free loop space of Σ)

Structures on $\mathcal{A}$:

- Goldman bracket $\Lambda^2 \mathcal{A} \to \mathcal{A}$



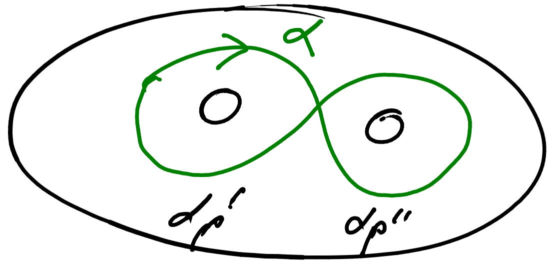
α, β transverse representatives

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \, (\alpha_p \beta_p)$$

do simple surgery at p.

local intersection # of p

- Turaev cobracket



$$\delta(\alpha) = \sum_{p \in \alpha \cap \alpha} \varepsilon_p \, \alpha_p' \wedge \alpha_p''$$

<u>Thm</u>

- (Goldman) $[\,,\,]$ is well defined and it is a Lie bracket
- (Turaev) δ is well defined (if one fixes a framing of $T\Sigma$) is a cobracket and a 1-cocycle

$$\delta : \mathcal{A} \to \Lambda^2 \mathcal{A}$$

<u>Cor</u>: $(\mathcal{A}, [\,,\,], \delta)$ is a Lie bialgebra

—— ∘ ——

actually, by weight filtration

Filter $k\pi$ by powers of aug ideal.

Consider $gr(\bar{\mathcal{A}})$ = cyclic words in $H_1(\Sigma)$

This is a Lie bialgebra with bracket $[\,,\,]_{low}$, cobracket δ_{low}

Remark:

$$\Sigma_{0,3}: \quad (\mathrm{Gr}\,\overline{\mathcal{A}},\ [\ ,\ ]_{low}) = \text{special derivations}$$

$$\Sigma_{g,1}: \quad (\mathrm{Gr}\,\overline{\mathcal{A}},\ [\ ,\ ]_{low}) = \mathrm{Der}^{\circ}\,\mathbb{L}(H)$$

Thm (AKKN)

$$(\overline{\mathcal{A}},\ [\ ,\ ],\ \delta) \cong \overset{\mathrm{Gp}\ \circlearrowright}{(\mathrm{gr}\,\overline{\mathcal{A}},\ [\ ,\ ]_{low},\ \delta_{low})}$$

torsor of isomorphisms

$$\mathrm{Lie}(\mathrm{Gp}) = \mathrm{krv}_{g,n}$$