(Plan) A stabilizer interpretation of double shuffle Lie algebras | MSRI

(joint w. H. Furusho. 1605.02838. accepted IMRN.)

29-04-17 (1)

30 — **(A) Background** (10:30)

- (1) Inclusions of LAs of grt type
- (2) cyclotomic variants. (10:50)

(11:00)

~~(B) The main result~~.

10 [(3) the Lie algebras dmr_0 / $dmr_0(N)$

5 [(B) THE MAIN RESULT] Lie algebraic constructions based on $outder^*(\mathbb{L}(X))^{\Gamma}$.

20 — 15 [

- (1) Lie algebraic constructions based on ...
- (2) $dmr_0(N)$ as a stabilizer (11:15)

5 [(3) Idea of proof. (11:20)

10 — **(C) Future directions**

- (1) Group ~~analog~~ version of main result.
- (2) Construction of Δ^{Betti}. (11:30)

(10^{30})

(ex of Periods: MZVs, cycloMZV)

(A) Background

MZVs = 2 types of relations: associator / double shuffle + reg. → schemes: Assoc / DMR

Schemes are torsors.

Relation between schemes & groups:

$\sum \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}$, $n_1 > \cdots > n_m$, $= \zeta(n_1 \ldots n_m)$

$(Sol(\omega_B, \omega_{DR}))_\lambda$ →(VB) $Assoc_\lambda(k)$ →(F) $DMR_\lambda(k)$ → $SolKV(k)$

$Aut^{\otimes}(\omega_{DR})$ ↻ $GRT_1(k)$ →(HF) $DMR_0(k)$ →(LS) $KRV(k)$

$\mathfrak{g}^{u}$ →(B) grt_1 ; dmr_0

$\mathbb{L}(\ldots)$; $grt_1 = im(outder^*(P_5)^{S_5} \to outder^*(P_4)^{S_4})$

	gp-torsor
GT	$outder^*(P_5)$, Σ_5
DMR	(1)
KV	... SolKV
GT-DMR	
GT-KV	
DMR-KV	(S.)

Cyclo MZVs. $\sum_{n_1 > \cdots > n_m > 0} \frac{\zeta_1^{n_1} \cdots \zeta_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} = L(k_1 \ldots k_m | \zeta_1 \ldots \zeta_m)$

$Psdist_\lambda(k) \to DMR_\lambda(N, k)$

↻ $GRT_1(N, k) \to DMR_0(N, k)$ ↻

$grt_1(N)$; $dmr_0(N)$

	gp-torsor
GT	ont?
DMR	(1)
GT-DMR	(2)

(10:50)

① Def. of the Lie algebra $\mathfrak{dmr}_0 / \mathfrak{dmr}_0(N)$

ⓐ The Lie algebra $(\mathfrak{L}(X), \langle\,\rangle)$

$X := \{x_0, x_\sigma \mid \sigma \in \mu_N\} \Rightarrow \mathfrak{L}(X)$ $\circlearrowleft \mu_N$

$\mathfrak{L}(X) \ni a \mapsto d_a \in \mathrm{Der}(\mathfrak{L}(X))^{\mu_N}$

$\begin{cases} x_0 \to 0 \\ x_1 \to [x_1, a] \end{cases}$

$\langle a,b\rangle := [a,b] + d_a(b) - d_b(a)$

so that $\left(\begin{matrix} (\mathfrak{L}(X), \langle\,\rangle) \to \mathrm{Der}(\mathfrak{L}(X))^{\mu_N} \\ a \longmapsto d_a \end{matrix} \right) =$ LA morphism

ⓑ Alg. material

$\mathbb{Q}\langle Y\rangle := \mathbb{Q}\langle X\rangle / \mathbb{Q}\langle X\rangle x_0 = \mathbb{Q}\langle y_{n,\sigma}\rangle$ ($n \geq 1$, $\sigma \in \mu_N$), $y_{n,\sigma} = x_0^{n-1} x_\sigma$

$\Delta_* \in \mathrm{Hom}(\mathbb{Q}\langle Y\rangle, \mathbb{Q}\langle Y\rangle^{\otimes 2})$

alg. map: $y_{n,\sigma} \to y_{n,\sigma} \otimes 1 + 1 \otimes y_{n,\sigma} + \sum y_{\ldots} \otimes y_{\ldots}$

$\mathbb{Q}\langle X\rangle \longrightarrow \mathbb{Q}\langle Y\rangle$

$\varphi \mapsto \varphi_* = \mathrm{proj}(\varphi) + \sum \frac{(-1)^{n-1}}{n} (\varphi \mid x_0^{n-1} x_1)(y_{11})^n$

ⓒ

$\mathfrak{dmr}_0(N) = \left\{ \varphi \in \mathfrak{L}(X) \;\middle|\; \begin{matrix} (\varphi \mid x_0) = (\varphi \mid x_1) = 0 \\ (\varphi \mid x_\sigma) = (\varphi \mid x_{\sigma^{-1}}) \\ \varphi_* \text{ is } \Delta_*\text{-primitive} \end{matrix} \right\} \subset (\mathfrak{L}(X), \langle\,\rangle)$ (LA)

11:00

} The main result.

③ LA Construction based on $\mathrm{outder}^*(\mathfrak{L}(X))^{\mu_N}$

fact 1 $(\mathfrak{L}(X), \langle\,\rangle) \simeq \mathrm{outder}^*(\mathfrak{L}(X))^{\mu_N} \oplus (\mathbb{Q}x_0 \oplus \mathbb{Q}x_1)$

Proof: $\left(\begin{matrix} \mathfrak{L}(X) \to \mathrm{der}^*(\mathfrak{L}(X))^{\mu_N} \to \mathrm{outder}^* \\ a \longrightarrow d_a \longrightarrow [d_a] \end{matrix} \right)$ = epi

$\Rightarrow \ker = \mathbb{Q}x_0 \oplus \mathbb{Q}x_1$

$\exists$ section (grading)

Fact 2: $\left(\begin{matrix} (\mathfrak{L}(X), \langle\,\rangle) \xrightarrow{\alpha} \mathrm{Der}(\mathfrak{L}(X)) \\ a \mapsto d_a \end{matrix} \right)$ (epi) = LA morph

Fact 3 $\left(\begin{matrix} (\mathfrak{L}(X), \langle\,\rangle) \xrightarrow{\beta} \mathrm{Der}(\mathfrak{L}(X)) \\ a \longmapsto d_a + \mathrm{ad}(a) \end{matrix} \right)$ = LA morph.

Proof: $[d_a + \mathrm{ad}\,a, d_b + \mathrm{ad}\,b] = d_{\langle ab\rangle} + \mathrm{ad}(d_a b - d_b a + [ab])$
$= d_{\langle ab\rangle} + \mathrm{ad}\,\langle ab\rangle.$ $\triangleright$

Set: $\left(\begin{matrix} (\mathfrak{L}(X), \langle\,\rangle) \xrightarrow{\lambda} \mathfrak{L}(X) \\ a \longmapsto a \end{matrix} \right)$

General framework | $\mathfrak{g}, \mathfrak{a}$ = 2 Lie algebras / α, β, λ = linear maps
s.t. $\beta_x = \alpha_x + [\lambda(x), -]$

$$\mathfrak{g} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathrm{Der}(\mathfrak{a}), \quad \lambda \downarrow \mathfrak{a}$$

(1) α = LA morph
(2) β = LA morph
(3) $\lambda[xy] = [\lambda x, \lambda y] + \alpha_x(\lambda y) - \alpha_y(\lambda x)$.

(rem: $12 \Rightarrow 3$, $13 \Rightarrow 2$, $23 \Rightarrow 1$)

$\mathfrak{g}$-mod, 2 algebras: $U(\mathfrak{a})_\alpha$, $U(\mathfrak{a})_\beta$
& bimodule: $U(\mathfrak{a})_{\beta\alpha}$ $\quad x \cdot m := \alpha_x(m) + \lambda(x) \cdot m$

$\ker(\alpha)$ & $\ker(\beta)$ are Lie subalgs of $\mathfrak{g}$ commuting if λ injective $\Rightarrow$ 2 commuting actions on $U(\mathfrak{a})_{\alpha\beta\alpha}$
$\ker(\alpha)$ = left mult. via $\lambda(x)$
$\ker(\beta)$ = right mult via $-\lambda(x)$.

If $\ker(\beta)$ is central $\Downarrow$

a $\mathfrak{g}$-module $U(\mathfrak{a}) / U(\mathfrak{a})\lambda(\ker_\beta)$

if $\ker(\alpha) \subset Z(\mathfrak{g})$, then $\mathfrak{g}$-action commu. w. left mult by $\lambda(\ker(\alpha))$

& $\ker\alpha$ is central

Prop 1: a $(\mathbb{L}(X), \langle,\rangle)$-module
$\mathbb{Q}\langle Y\rangle = \mathbb{Q}\langle X\rangle / \mathbb{Q}\langle X\rangle \cdot x_0$,
$\psi \cdot \bar{a} = \overline{d_\psi(a) + \psi \cdot a}$

action of $\psi \in \mathbb{L}(X)$ commu. w. (left mult by x_1) $= T \in \mathrm{End}(\mathbb{Q}\langle Y\rangle)$

Prop 2: $\exists$ characters of $(\mathbb{L}(X), \langle,\rangle)$
$\chi_n : (\mathbb{L}(X), \langle,\rangle) \to \mathbb{Q}$
$a \mapsto (a | x_0^{n-1} x_1)$.

Pf: look at weights. $\square$

Prop 3: $\exists$ a $(\mathbb{L}(X), \langle,\rangle)$-module structure
$\mathbb{Q}\langle Y\rangle = \mathbb{Q}\langle X\rangle / \mathbb{Q}\langle X\rangle x_0$
$$\psi * \bar{a} := \psi \cdot \bar{a} + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \chi_n(\psi) \cdot T^n a$$

Pf follows from Props 1 & 2. $\square$

Thm: $\mathrm{dmr}_0(\mathbb{Q}) \oplus (\mathbb{Q}x_0 \oplus \mathbb{Q}x_1) = \mathrm{stab}(\Delta_*) = \{\psi \in (\mathbb{L}(X), \langle,\rangle) \mid \Delta_* \circ (\psi * -) = ((\psi * -) \otimes 1 + 1 \otimes (\psi * -)) \circ \Delta_*\}$

Pf: based on Racinet's techniques.

11:20

(C) Construction of $\Delta_\infty^{\text{Beltr}}$.

$\exp(\mathcal{L}(x), \langle\,\rangle)$

subexp $\subset$ $\cup$

$\exp(\mathrm{Stab}(\Delta_*)) \circlearrowright \mathrm{OMR}_\lambda$

$= \mathrm{Stab}(\Delta_*)$; action is pdt.

$\Rightarrow$

$\{g^{-1} \cdot \Delta_* \mid g \in \mathrm{OMR}_\lambda\} = \text{Belt.} = \Delta_*^{\text{Belti}}$

$\Downarrow$

$\mathrm{OMR}_\lambda = \{ g \in \exp(\mathcal{L}(x), \langle , \rangle) \mid g \cdot \Delta_* = \Delta_*^{B} \}$

construction of "right group".

$\Phi : G \to \mathrm{Stab}(\Delta_*^{B})$.

2017.03.29

Benjamin Enriquez

A stabilizer interpretation of double shuffle Lie algebras

Joint with Furusho (on Arxiv)

Assoc	Formal MZVs (solns of double shuffle)	SolKV
$\circlearrowleft$	$\circlearrowleft$	$\circlearrowleft$
GRT_1 $\hookrightarrow$	DS $\hookrightarrow$	KRV
$\mathfrak{grt}_1$ $\hookrightarrow$	$\mathfrak{ds}$ $\hookrightarrow$	$\mathfrak{krv}$

$$\mathfrak{krv} = \left\{ D \in \mathrm{Der}\,\mathbb{L}(x_0, x_1) : D(x_0) = [x_0, *],\ D(x_1) = [x_1, *],\ D(x_0 + x_1) = 0,\ \mathrm{div}\,D = \mathrm{tr}\left(f(x_0+x_1) - f(x_0) - f(x_1)\right) \text{ some } f \in k[t]. \right\}$$

Ihara:

inertia condition: $t_{jk} \mapsto [t_{jk}, *]$

$$(\mathrm{Out}^* \mathfrak{p}_n)^{S_n} \xrightarrow{\simeq} (\mathrm{Out}^* \mathfrak{p}_{n-1})^{S_{n-1}} \xrightarrow{\simeq} \cdots \hookrightarrow (\mathrm{Out}^* \mathfrak{p}_4)^{S_4}$$

↑ rest. isoms.

$\mathfrak{p}_n$ = Lie alg of Pure braid gp of $\mathbb{P}^1$

$$\underline{\mathfrak{grt}_1} := \text{image}\left\{ \left(\mathrm{Out}^* \mathfrak{f}_5\right)^{S_5} \hookrightarrow \mathrm{Out}^* \mathfrak{f}_4 \right\}$$

<u>Double Shuffle Lie algebra</u> (Racinet)

Here treat the cyclotomic analogue.

$\rightsquigarrow \mathfrak{ds}(N) \longleftrightarrow$ cyclotomic analogues of MZVs.

$N \geq 1$ standard case is $N = 1$.

$X = \{ x_0, x_\sigma : \sigma \in \mu_N \} \circlearrowleft \mu_N$

$(\mathbb{L}(X), \langle , \rangle)$ Ihara bracket

$$\langle a, b \rangle = [a, b] + d_a(b) - d_b(a)$$

where $\quad d_a \in \mathrm{Der}\, \mathbb{L}(X)^{\mu_N}$

satisfies

$$x_0 \longmapsto 0, \quad x_1 \longmapsto [x_1, a]$$

$x_\sigma \mapsto [x_\sigma, a]$ invariance

There is a morphism

$$(\mathbb{L}(X), \langle , \rangle) \to \mathrm{Der}\, \mathbb{L}(X)^{\mu_N}$$

$$a \longmapsto d_a$$

$$\rightsquigarrow \quad \mathfrak{ds}(N) \hookrightarrow (\mathbb{L}(X), \langle , \rangle)$$

$\mathbb{Q}\langle X\rangle$ = free assoc algebra

alg morphism

$$\mathbb{Q}\langle X\rangle / \mathbb{Q}\langle X\rangle x_0 \cong \mathbb{Q}\langle Y\rangle \xrightarrow{\Delta_*} \mathbb{Q}\langle Y\rangle^{\otimes 2}$$

free assoc alg on $y_{n\sigma}$ $\qquad y_{n,\sigma} = x_0^{n-1} x_\sigma$

$\sigma \in \mu_N \quad n \geq 1$

$$\Delta_*(y_{n\sigma}) = y_{n\sigma} \otimes 1 + 1 \otimes y_{n\sigma} + \sum_{\substack{n'+n''=n \\ n',n''>0 \\ \sigma'\cdot\sigma''=\sigma}} y_{n'\sigma'} \otimes y_{n''\sigma''}$$

$$\mathbb{Q}\langle X\rangle \longrightarrow \mathbb{Q}\langle Y\rangle$$

$$\psi \longmapsto \psi_* = \mathrm{proj}(\psi) + \sum \frac{(-1)^{n-1}}{n} (\psi \mid x_0^{n-1} x_1)\, y_n^n$$

Def:

$ds(N) = \{ \psi \in \mathbb{L}(X) :$ ψ_* is Δ_*-primitive + cond^ns above $\}$

$\hookrightarrow (\mathbb{L}(X), \langle\,,\rangle)$

Subalgebra (Racinet)

$x_0 \mapsto [x_0, *]$

$/ x_\sigma \mapsto [x_\sigma, *]$

Facts:

① $(\mathbb{L}(X), \langle , \rangle) \cong \text{Out-Der}^* \, \mathbb{L}(X)^{\mu_N} \oplus \underbrace{\mathbb{Q}x_0 \oplus \mathbb{Q}x_1}_{\text{abelian Lie alg}}$

Proof.

$$(\mathbb{L}(X), \langle , \rangle) \to (\text{Der}^* \, \mathbb{L}(X))^{\mu_N} \to (\text{Out Der}^* \, \mathbb{L}(X))^{\mu_N}$$

$$a \longmapsto d_a \longmapsto [d_a]$$

surjective. Kernel is $\mathbb{Q}x_0 \oplus \mathbb{Q}x_1$

Use graded splitting (why LA homom?) □

② There is LA homom

$$(\mathbb{L}(X), \langle , \rangle) \xrightarrow{\alpha} \text{Der} \, \mathbb{L}(X)$$

$$a \longmapsto d_a$$

③ There is LA homom

$$(\mathbb{L}(X), \langle , \rangle) \xrightarrow{\beta} \text{Der} \, \mathbb{L}(X)$$

$$a \longmapsto d_a + (a, -)$$

— ad_a, inner

Generalization:

$\mathfrak{g}$, $\mathfrak{a}$ Lie algebras

$$\mathfrak{g} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \mathrm{Der}(\mathfrak{a})$$

α, β LA homoms

linear $\to$ $\lambda \downarrow$

$\mathfrak{a}$

$$\lambda([x,y]) = [\lambda x, \lambda y] + \alpha_x(\lambda y) - \alpha_y(\lambda x)$$

$$\beta_x = \alpha_x + [\lambda(x), -]$$

(A) In $\mathfrak{g}$-module, have

(i) $U(\mathfrak{a})_\alpha = U\mathfrak{a}_\alpha$?

$U(\mathfrak{a})_\beta =$