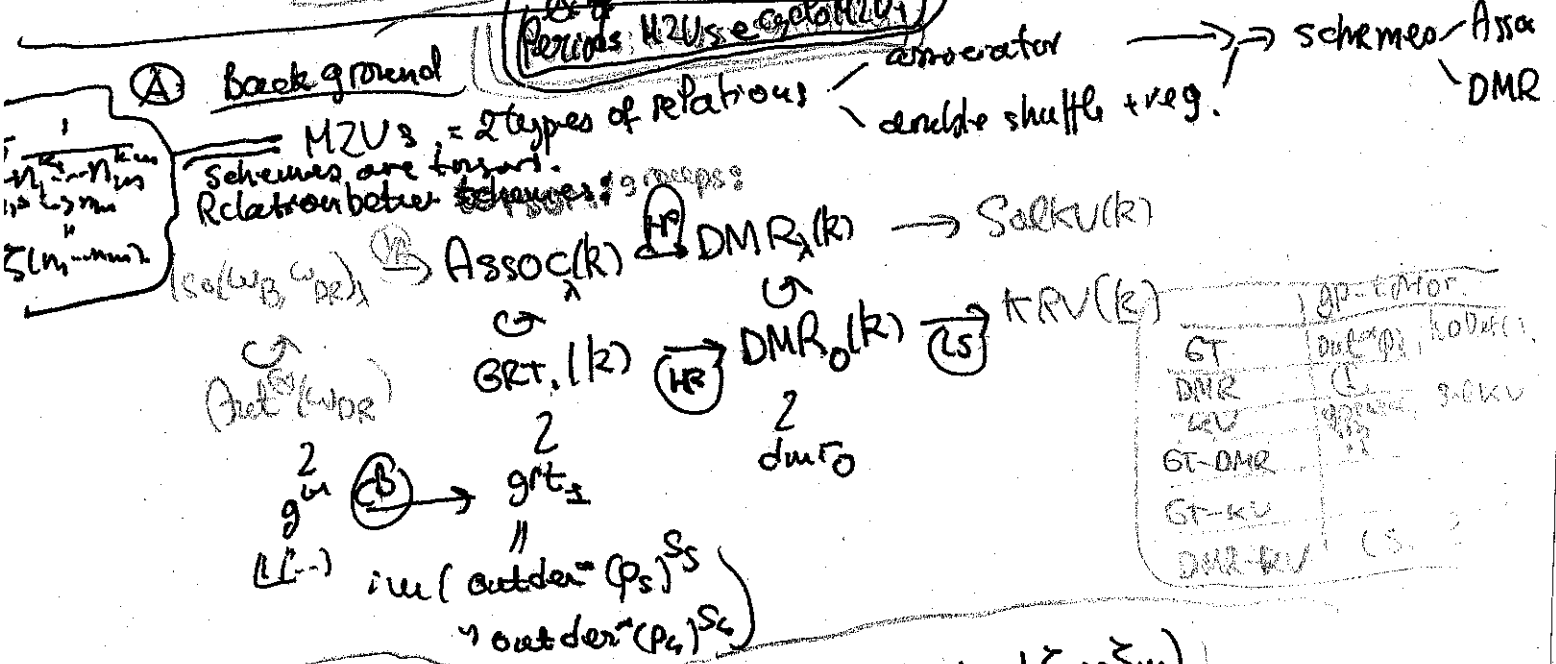
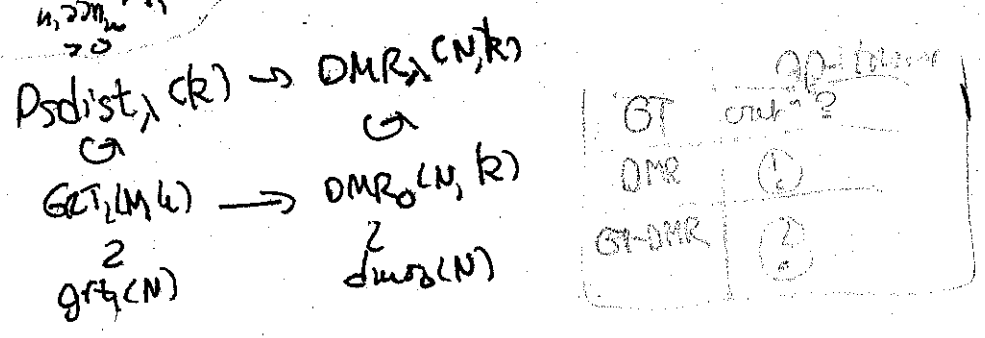


(Plan) A stabilizer interpretation of double shuffle Lie algebras | MR1
 (joint w. H. Furusho. 1605.02838. accepted MRN. 29-04-17) ①

- ① Background (10:30)
- ① Inclusions of LAs of grt type
 - ② cyclotomic variants. (10:50)
- ② ~~Background~~ (11:00)
- ③ the Lie algebras $\text{dms}_0 / \text{dms}_0(N)$
- ④ MAIN RESULT (11:15)
- ① Lie algebraic constructions based on $\text{outer}^*(\mathbb{L}(X))$.
 - ② $\text{dms}_0(N)$ as a stabilizer
 - ③ idea of proof. (11:20)
- ⑤ Future directions (11:30)
- ① Group ~~action~~ version of main result.
 - ② Construction of Δ_{Betti} . (10:30)



Cyelo MZVs. $\sum_{k_1, \dots, k_m} \frac{1}{h_{k_1} \dots h_{k_m}} s(\mathbb{L}(k_1, \dots, k_m) | \xi_1, \dots, \xi_m)$



10:50

~~Def. of the Lie algebra~~

Def. of the Lie algebra $\mathfrak{dun}_0 / \mathfrak{dun}_0(N)$

(a) The Lie algebra $(U(X), \langle, \rangle)$

$X := \{x_0, x_1, \dots, x_n\} \Rightarrow U(X)$

$U(X) \ni a \mapsto d_a \in \text{Der}(U(X))^{\text{un}}$

$\begin{cases} x_0 \rightarrow 0 \\ x_i \rightarrow [e_i, a] \end{cases}$

$\langle a, b \rangle := \langle a, b \rangle + d_a(b) - d_b(a)$

so that $(U(X), \langle, \rangle) \rightarrow \text{Der}(U(X))^{\text{un}} = \text{LA morphism}$
 $a \mapsto d_a$

(b) Algebra material

$\mathbb{Q}(Y) := \mathbb{Q}(X) / \mathbb{Q}(X)_{x_0} \cong \mathbb{Q}(y_1, \dots, y_n)$

$\Delta_* \in \text{Hom}(\mathbb{Q}(Y), \mathbb{Q}(Y))$

$\mathbb{Q}(X) \rightarrow \mathbb{Q}(Y)$
 $\downarrow \phi$
 $\phi_* = \text{proj}(\phi) + \sum \frac{(-1)^{n-1}}{n} (\phi(x_0^{n-1} x_i)) (y_i)^n$

(c) $\mathfrak{dun}_0(N) = \{ \psi \in U(X) \mid \begin{matrix} \psi(x_0) = \psi(x_1) = 0 \\ \psi(x_0) = \psi(x_0^{-1}) \\ \psi_* \text{ is } \Delta_*\text{-primitive} \end{matrix} \}$

The main result: LA

Construction based on $\text{outer}^*(U(X))$

Fact 1: $(U(X), \langle, \rangle) \cong \text{outer}^*(U(X))^{\text{un}} \oplus (\phi_{x_0} = \phi_{x_1})$

Proof: $(U(X) \rightarrow \text{Der}^*(U(X))^{\text{un}} \rightarrow \text{outer}^*)$
 $(a \mapsto d_a \mapsto [d_a])$
 epi
 $\ker = \phi_{x_0} = \phi_{x_1}$
 section (grading)

Fact 2: $(U(X), \langle, \rangle) \xrightarrow{\text{epi}} \text{Der}(U(X)) = \text{LA morph}$
 $a \mapsto d_a$

Fact 3: $(U(X), \langle, \rangle) \xrightarrow{\text{epi}} \text{Der}(U(X)) = \text{LA morph.}$
 $a \mapsto d_a + \text{ad}(a)$

Proof: $[d_a + \text{ad}(a), d_b + \text{ad}(b)] = d_a(b) + \text{ad}(a)(b - d_b(a) + \langle a, b \rangle)$
 $= d_a(b) + \text{ad}(a)(\langle a, b \rangle)$

Set: $(U(X), \langle, \rangle) \xrightarrow{\lambda} U(X)$
 $a \mapsto a$

General Framework

g, α, β 2 Lie algebras / α, β, λ 3 derivations.

(3)

$g \xrightarrow[\beta]{\alpha} \text{Der}(A)$
 $\lambda \downarrow$
 A

- (1) $\alpha = \text{LA morph}$
- (2) $\beta = \text{LA morph}$
- (3) $\lambda[x, y] = [\alpha(x), \beta(y)] + \alpha(x)(\beta(y)) - \alpha(y)(\beta(x))$.

(rem: $1 \Rightarrow 3, 3 \Rightarrow 2, 2 \Rightarrow 1$)

~~... ..~~

in g -mod, 2 algebras: $U(\alpha), U(\beta)$
 + bimodule: $U(\alpha) \otimes U(\beta)$ $x \cdot m := \alpha(x)(m) + \lambda(x) \cdot m$

$\ker(\alpha) \oplus \ker(\beta)$ are Lie subalgs of g
 commuting if λ injective

2 commuting actions on $U(\alpha) \otimes U(\beta)$
 $\ker(\alpha) = \text{left mult. via } \lambda(\alpha)$
 $\ker(\beta) = \text{right mult via } -\lambda(\alpha)$

Prop 1

a $(U(x), \mathcal{G})$ -module
 $\mathcal{Q}(Y) = \mathcal{Q}(X) / \mathcal{Q}(X) \cdot x_0$
 $\psi \cdot \bar{a} = \overline{\psi(a) + \psi \cdot a}$

action of $\psi \in U(x)$
 comm. w. (left mult by x_1)
 \parallel
 $\Gamma \in \text{End } \mathcal{Q}(Y)$

if $\ker(\beta)$ is central

a g -module $U(\alpha) / U(\alpha) \lambda(\ker(\beta))$

if $\ker(\alpha) \subset Z(g)$, then
 g -action comm. w.
 left mult by $\lambda(\ker(\alpha))$

Prop 2

\exists characters of $(U(x), \mathcal{G})$
 $\chi: (U(x), \mathcal{G}) \rightarrow \mathbb{Q}$
 $a \mapsto (a | x_0^{n-1} x_1)$

pf: look at weights. 0

Prop 3

\exists a $(U(x), \mathcal{G})$ -module shwn
 $\mathcal{Q}(Y) = \mathcal{Q}(X) / \mathcal{Q}(X) x_0$
 $\psi * \bar{a} := \psi \circ \bar{a} + \sum_{k \geq 1} \frac{(-1)^k}{k} \chi_k(x) \cdot T^k a$

pf follows from Prop 1 & 2. \square

Thm

$\text{dim}_0(CN) \oplus (\mathcal{Q} x_0 \oplus \mathcal{Q} x_1) = \text{stab}(\Delta_*)$
 \parallel
 $\{ \psi \in (U(x), \mathcal{G}) \}$

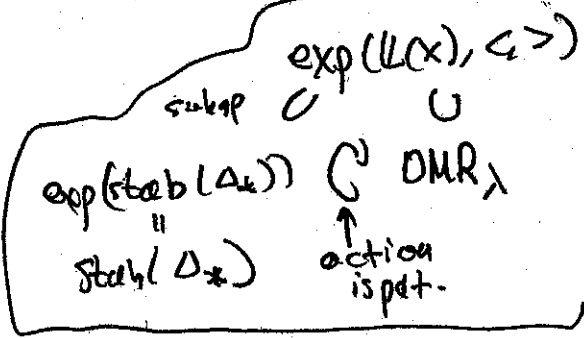
$\Delta_{\psi} \circ (\psi \dashv -) =$
 $(\psi \dashv -) \circ (\psi \dashv -) \circ \Delta_{\psi}$

pf: based on Racinet's techniques.

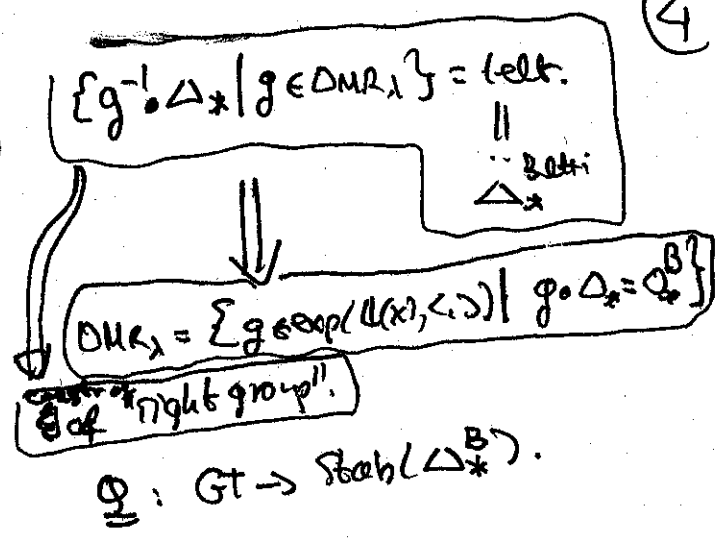
11:20

~~...~~

③ Construction of Δ^B



\Rightarrow



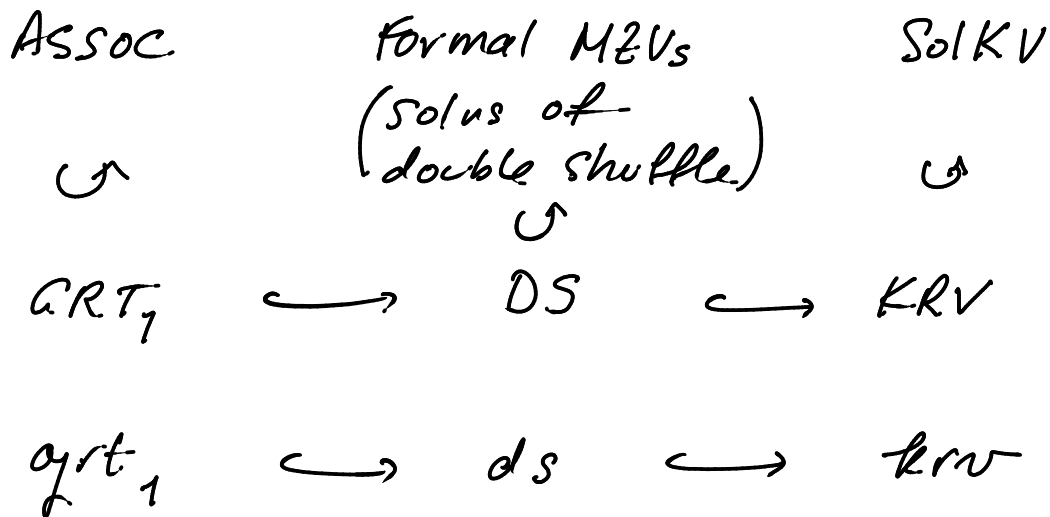
④

2017.03.29

Benjamin Enriquez

A stabilizer interpretation of double shuffle Lie algebras

Joint with Furusho (on Arxiv)



$$krv = \left\{ \begin{array}{l} D \in \text{Der } \mathbb{L}(x_0, x_1) : D(x_0) = [x_0, *] \\ D(x_1) = [x_1, *], D(x_0 + x_1) = 0 \\ \text{div } D = \text{tr} (f(x_0 + x_1) - f(x_0) - f(x_1)) \\ \text{some } f \in k[t]. \end{array} \right.$$

Ihara:

$$(\text{Out}^* \mathfrak{p}_n)^{S_n} \xrightarrow{\cong} (\text{Out}^* \mathfrak{p}_{n-1})^{S_{n-1}} \dots \hookrightarrow (\text{Out}^* \mathfrak{p}_4)^{S_4}$$

inertia condition: $t_{j,k} \mapsto [t_{j,k}, *]$

↑ rest isom.

$\mathfrak{p}_n =$ Lie alg of pure braid gp of \mathbb{P}^1

$$\underline{grt}_1 := \text{image} \left\{ \left(\text{Out}^* \mathbb{P}^5 \right)^{S_5} \hookrightarrow \text{Out}^* \mathbb{P}^4 \right\}$$

Double Shuffle Lie algebra (Racinet)

Here treat the cyclotomic analogue.

$\leadsto ds(N) \leftrightarrow$ cyclotomic analogues of
MZVs.
 $N \geq 1$ standard case is $N=1$.

$$X = \{ x_\sigma, x_\sigma : \sigma \in \mu_N \} \curvearrowright \mu_N$$

$(\mathbb{L}(X), \langle, \rangle)$ Ihara bracket

$$\langle a, b \rangle = [a, b] + d_a(b) - d_b(a)$$

where

$$d_a \in \text{Der } \mathbb{L}(X)^{\mu_N}$$

satisfies

$$x_0 \mapsto 0, \quad x_1 \mapsto [x_1, a]$$

$$x_\sigma \mapsto [x_\sigma, a] \text{ invariance}$$

There is a morphism

$$(\mathbb{L}(X), \langle, \rangle) \rightarrow \text{Der } \mathbb{L}(X)^{\mu_N}$$

$$a \mapsto d_a$$

$$\leadsto ds(N) \hookrightarrow (\mathbb{L}(X), \langle, \rangle)$$

$\mathbb{Q}\langle X \rangle =$ free assoc algebra alg morphism

$$\mathbb{Q}\langle X \rangle / \mathbb{Q}\langle X \rangle x_0 \cong \mathbb{Q}\langle Y \rangle \xrightarrow{\Delta_*} \mathbb{Q}\langle Y \rangle^{\otimes 2}$$

free assoc
alg on y_n

$$y_{n,\sigma} = x_0^{n-1} x_\sigma$$

$\sigma \in \mu_N \quad n \geq 1$

$$\Delta_*(y_{n\sigma}) = y_{n\sigma} \otimes 1 + 1 \otimes y_{n\sigma}$$

$$+ \sum_{n'+n''=n} y_{n'\sigma'} \otimes y_{n''\sigma''}$$

$$n', n'' > 0$$

$$\sigma' \cdot \sigma'' = \sigma$$

$$\mathbb{Q}\langle X \rangle \longrightarrow \mathbb{Q}\langle Y \rangle$$

$$\psi \longmapsto \psi_* = \text{proj}(\psi) + \sum \frac{(\psi)}{n} x_0^{n-1}$$

$$(\psi x_0^{n-1} x_i) y_n^n$$

Def:

$$\mathcal{ds}(N) = \left\{ \psi \in \mathbb{L}(X) : \begin{array}{l} \psi_* \text{ is } \Delta_* \text{-primitive} \\ + \text{conds above} \end{array} \right\}$$

$$\hookrightarrow (\mathbb{L}(X), \langle, \rangle)$$

Subalgebra (Racinet)

Facts:

$$x_0 \mapsto [x_0, *]$$

$$x_1 \mapsto [x_1, *]$$

$$\textcircled{1} (\mathbb{L}(X), \langle, \rangle) \cong \text{Out Der}^* \mathbb{L}(X)^{\text{MN}}$$

$$\oplus \underbrace{\mathbb{Q}x_0 \oplus \mathbb{Q}x_1}_{\text{abelian Lie alg}}$$

abelian Lie alg

Proof.

$$(\mathbb{L}(X), \langle, \rangle) \rightarrow (\text{Der}^* \mathbb{L}(X))^{\text{MN}} \rightarrow (\text{Out Der}^* \mathbb{L}(X))^{\text{MN}}$$

$$a \longmapsto d_a \longmapsto [d_a]$$

surjective. Kernel is $\mathbb{Q}x_0 \oplus \mathbb{Q}x_1$

Use graded splitting (why LA homom?) \square

$\textcircled{2}$ There is LA homom

$$(\mathbb{L}(X), \langle, \rangle) \xrightarrow{\alpha} \text{Der } \mathbb{L}(X)$$

$$a \longmapsto d_a$$

$\textcircled{3}$ There is LA homom

$$(\mathbb{L}(X), \langle, \rangle) \xrightarrow{\beta} \text{Der } \mathbb{L}(X)$$

$$a \longmapsto d_a + (a, -) \quad \text{--- } ad_a, \text{ inner}$$

Generalization:

$\mathfrak{g}, \mathfrak{a}$. Lie algebras

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow[\beta]{\alpha} & \text{Der}(\mathfrak{a}) \\ \downarrow & & \\ \mathfrak{a} & & \end{array} \quad \begin{array}{l} \text{linear} \rightarrow \lambda \\ \alpha, \beta \text{ LA homoms} \end{array}$$

$$\lambda([x, y]) = [\lambda x, \alpha y] + \alpha_x(\lambda y) - \alpha_y(\lambda x)$$

$$\beta_x = \alpha_x + [\lambda(x), -]$$

(A) In \mathfrak{g} -module, have

$$(1) U(\mathfrak{a})_{\alpha} = U \mathfrak{a} \quad ?$$

$$U(\mathfrak{a})_{\beta} =$$