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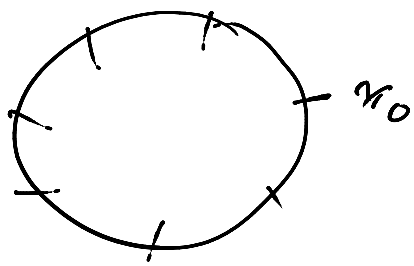
$\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ as operad

Joint: Thomas Willwacher

Fact $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ smooth manifold / variety

operad structure



$$S \cup T = \{0, \dots, n+1\}$$

$$\overline{\mathcal{M}}_{0,S} \times \overline{\mathcal{M}}_{0,T} \subseteq \overline{\mathcal{M}}_{0,n+1}$$

$$\mathcal{M}_{0,4}(\mathbb{R}) = \mathbb{P}^1(\mathbb{R}) \cong S^1$$

cross ratio

$\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ smooth non oriented variety

Betti numbers: $b_1 = 0$, $b_4 = 2$

+ 2-torsion

Explain: $\mathbb{Q}$-htpy type of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$

Motivation from representation theory

Thm (2003) (Davis–Januszkiewicz–Scott)

$$\overline{M}_{0,n+1}(\mathbb{R}) = K(PCacti_n, 1)$$

$PCacti_n$ = pure Cacti group.

PB_n = pure braid group

$U_q\mathfrak{g}$ = a quantum gp

$Rep(U_q\mathfrak{g})$ is a braided $\otimes$ category

$R_{V,W} : V\otimes W \xrightarrow{\sim} W\otimes V$ braiding.

$$\begin{tikzcd}
 & U\otimes V\otimes W \arrow[dl, "R_{U,V}\otimes 1"'] \arrow[dr, "R_{U,V\otimes W}"] & \\
V\otimes U\otimes W \arrow[rr, "1\otimes R_{U,W}"'] & & V\otimes W\otimes U
\end{tikzcd}$$

use assoc or assume assoc $\otimes$.

Implies that B_n = braid group acts on $V^{\otimes n}$

$$\sigma_i \mapsto 1\otimes R_{V_i,V_{i+1}}\otimes 1$$

Sim: $PB_n \curvearrowright V_1 \otimes \cdots \otimes V_n$

<u>Definition</u> (Drinfeld) $\mathcal{C}$ is a coboundary category if there is a tensor product $\otimes$ in $\mathcal{C}$ and

$$c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$$

s.t. $\quad c_{V,W} \circ c_{W,V} = id$

Octagon relations:

$$\begin{tikzcd}
& U \otimes V \otimes W \arrow[dl, "c_{U,V}"'] \arrow[dr, "c_{V,W}"] & \\
V \otimes U \otimes W \arrow[dr, "c_{V\otimes U, W}"'] & & U \otimes W \otimes V \arrow[dl, "c_{U, W\otimes V}"] \\
& W \otimes V \otimes U &
\end{tikzcd}$$

$\otimes$ associative <u>OR</u> use associator.

$Cacti_n$ is group generated by all permutations of factors.

$$P\,Cacti_n = \ker\{Cacti_n \to \Sigma_n\}$$

Thm: (Kamnitzer, Henriques)

Cact$_n$ generated by

$c_{p,q}$: $V_1 \otimes \cdots \otimes V_p \otimes \cdots \otimes V_q \otimes \cdots \otimes V_n$

$\to V \otimes \cdots \otimes V_q \otimes \cdots \otimes V_p \otimes \cdots \otimes V_n$

subject to

intervals.

$$c_{p,q}\, c_{rs} = \begin{cases} c_{rs}\, c_{pq} & [p,q] \cap [r,s] = \phi \\ c_{mn}\, c_{pq} & [p,q] \supset [r,s] \end{cases}$$

Example (Drinfeld, Kamnitzer–Heinz)

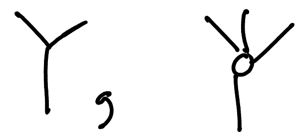
$\mathrm{Rep}(U_q\, \mathfrak{g}) \xrightarrow{q \to 0}$ is a coboundary cat

Crystals : M a set (numbers the basis of reps)

ex: $\mathfrak{g} = sl_2$

Summary of algebraic results on operad $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$

$H_\bullet(\ ) : \overline{\mathcal{M}}_{0,n+1} \longrightarrow$ quadratic operad generated by



mult, ternary Lie bracket

Etingof, ...

Chains

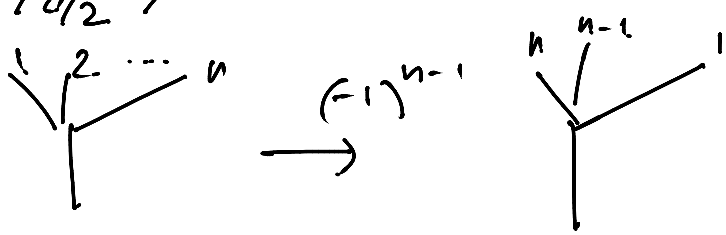
sats: Jacobi, canon assoc

Not formal

operad Koszul dual to

$\langle \mathrm{Assoc}/\mathbb{Z}/2 \rangle$

τ :



<u>Theorem</u> (A.Kh, Willwacher)

$\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is a rational $K(\pi,1)$

equiv: $H^\bullet(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}))$ is a quadratic Koszul algebra. Corresp Lie algebra is

$\mathbb{L}(\nu_{ijk} : [\nu_{ijk}, \nu_{pqi} + \nu_{pqj} + \nu_{pqk}] = 0$

$[\nu_{ijk}, \nu_{pqr}] = 0$ disjt $)$

Embedding

$$\mathfrak{t}_{odd}(n) \hookrightarrow \text{pure braid Lie alg}$$

$$v_{ijk} \longmapsto [t_{ij}, t_{jk}]$$