QUOTIENTS OF THE LIE LIE ALGEBRA

JIM CONANT

Abstract. We study two quotients of the Lie Lie algebra (the Lie algebra of symplectic derivations of the free Lie algebra), namely the abelianization and the the quotient by the Lie algebra generated by degree 1 elements. The abelianization has a very close connection to the homology of groups of automorphism groups of free groups, whereas the second is the so-called "Johnson cokernel," the cokernel of the Johnson homomorphism defined for mapping class groups of punctured surfaces

This work is partially joint with Karen Vogtmann and Martin Kassabov. Different parts are joint with different people and some of it is just by myself. So what is the Lie Lie algebra? It's the symplectic derivation algebra that we've seen before.

Consider $Der_\omega(\mathbb{L}(V))$, derivations of the free Lie algebra $\mathbb{L}(V)$ on a finite dimensional symplectic vector space V which kill the symplectic element ω. If $p_1, \ldots, p_n, q_1, \ldots, q_n$ is a symplectic basis for V, then $\omega = \sum_{i=1}^n [p_i, q_i]$. The derivations such that $D(\omega) = 0$ form the Lie Lie algebra.

Previously, in the conference, for example in Alekseev's talk, V was just 2 dimensional, since he was interested in the elliptic case. For us, we're mostly interested in the stable case, which means we're going to think of V as being very high dimensional.

So our derivations X, satisfy $X(\omega) = 0$. In the elliptic case, this just means that X kills the bracket of the two generators, whereas in general you actually have to kill the sum of the brackets of the generators.

Let's review the graphical interpretation of the Lie algebra, which we denote $D(V)$ for short. $D(V)$ is the k-span of unitrivalent trees (where k is a characteristic 0 field) where the trees have a cyclic ordering at every trivalent vertex and univalent vertices are labeled by vector space elements. There are three relations, AS, IHX and multilinearity:

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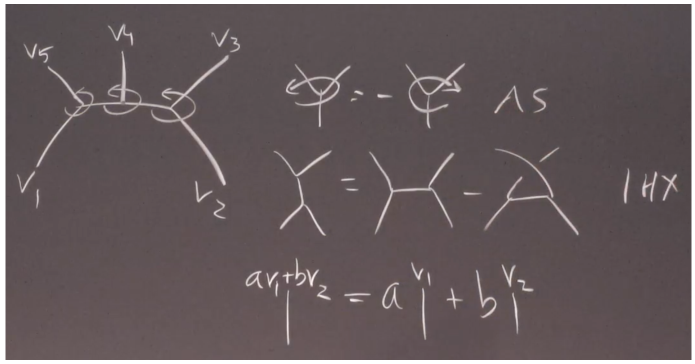
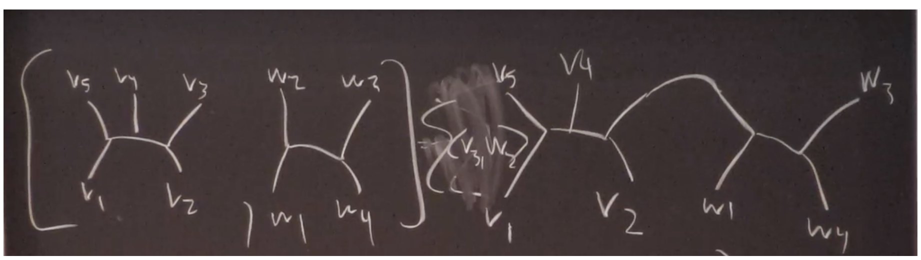


Multilinearity is the bottom right relation: if $av_1 + bv_2$ labels a univalent vertex, with $a, b \in k$, $v_1, v_2 \in V$, we can pull it apart as shown. Also, if you think of your trees in the plane, then the cyclic order is induced by the planar orientation, so a lot of times you don't even bother to write that into the picture.

This is actually the way I like to think about this symplectic derivation Lie algebra. There is a nice graphical interpretation of what the bracket is. The bracket between two trees is defined by summing over all ways of gluing a univalent vertex from the first tree onto a univalent vertex from the second tree. The labels of the two univalent vertices that are glued get contracted by the symplectic form. In the following picture I've shown one example of two vertices being joined, and then the sum sign is to indicate that this should be done for all pairs of univalent vertices.



The theorem is that there is an isomorphism from $D(V)$ to this graphical thing, and under that isomorphism the bracket goes to this graphically defined bracket.

What's so useful about this Lie algebra? Let's look at $D^+(V)$, the degree > 0 part, which means there exists at least one trivalent vertex. D^+ has a couple different applications. The abelianization $D^+/[D^+, D^+]$, this actually gives rise to homology classes for the outer automorphism group of the free group. This is one reason I came to be interested in this Lie algebra. It also has another application: it's connected to the mapping class group $Mod(g, 1)$, the mapping class group of a genus g surface with one boundary component $\Sigma_{g,1}$. There's a filtration of the mapping class group, due to Johnson, defined as follows. You have a map:

$$Mod(g, 1) \to Aut(\pi_1(\Sigma_{g,1}))$$

defined since an automorphism of the surface induces a self map on π_1 of that surface (which is just a free group here.) This induces a map

$$Mod(g,1) \to Aut(\pi_1(\Sigma_{g,1})/LCS_{n+1}(\pi_1(\Sigma_{g,1}))),$$

where LCS_{n+1} is the $n+1$st term in the lower central series. The kernel of this map is $\mathbb{J}_n$ is the (nth term in the Johnson filtration. For example, when $n=1$, you're dividing by the commutator subgroup, so $\mathbb{J}_1$ is the kernel of the map to the first homology of the surface. So you are looking at mapping classes that act trivially on first homology. I.e. $\mathbb{J}_1$ is the Torelli group.

Now I can form the associated graded, $\mathfrak{j} = \oplus_n(\mathbb{J}_n/\mathbb{J}_{n+1}) \otimes k$. It turns out that $\{\mathbb{J}_n\}$ is a central filtration, so $\mathbb{J}_n/\mathbb{J}_{n+1}$ are abelian groups, and we can tensor with the field k. The group commutator induces a Lie bracket on $\mathfrak{j}$. So $\mathfrak{j}$ is a Lie algebra. There is a map, called the Johnson homomorphism, which embeds this Lie algebra inside the symplectic derivation algebra, where V is the first homology of the surface.

The Johnson homomorphism is denoted τ_g:

$$\tau_g \colon \mathfrak{j}_g \to D^+(H_1(\Sigma_{g,1})).$$

(We've also indicated $\mathfrak{j}$'s dependence on g as well.)

Here is a brief sketch of the definition. Suppose $\varphi \in \mathbb{J}_n$, then $\varphi(x) = x \cdot \psi_x$m where ψ_x is a commutator of degree $n+1$. Look at $\sum_{b\in \text{basis}} b \otimes \psi_b$, where I think of ψ_b as being an element of the free Lie algebra as opposed to a group commutator. Let $V = H_1(\Sigma_{g,1})$. I can think of this as being inside $V \otimes \mathbb{L}(V)$, and V is naturally isomorphic to V^*, because I have a symplectic form. Then $V^* \otimes \mathbb{L}(V)$ is isomorphic to derivations of $\mathbb{L}(V)$, because a derivation of a free Lie algebra is determined by what it does to the generators. That's not the symplectic derivations, but the fact that your mapping class preserves the boundary, if you trace through the definition, means that the derivation kills the symplectic element.

One thing that's interesting is to try to determine the image of the Johnson homomorphism. This is where what I'm saying is related to the topic of this conference, although I don't know the details. If you look at $D^+(V)/im(\tau) = C$, this is called the Johnson Cokernel. Somehow the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is related to this cokernel, and there is an embedding

$$\mathbb{L}(\sigma_3, \sigma_5, \dots) \hookrightarrow C.$$

I know this embedding exists, but I don't know the full story.

We are interested in C as a vector space, or more precisely, as an $SP(V)$-module.

Theorem 1 (Hain)**.** *$im(\tau)$ is generated as a Lie algebra by the degree 1 part of $D^+(V)$, i.e. by labeled tripods.*

So now the Johnson cokernel can be defined completely algebraically as $C = D^+(V)/\langle\text{tripods}\rangle$. Note $C \twoheadrightarrow D^+/[D^+/D^+]$ in degree ≥ 2. So the abelianization gives us information about the cokernel. It turns out that it is enormously complicated and that we don't fully understand it. To paraphrase a certain politician, "Who knew that it could be this complicated?" Sad!

It turns out there is a rank that can be defined which came out of my work with Kassabov and Vogtmann. It is an additional grading of $D^+_{ab} \cong \oplus_{r\geq 1} D^+_{ab}[r]$.

Rank 1: (Morita).

$$D^+_{ab}[1] \oplus_{n\geq 1} S^{2n+1}(V)$$

Morita conjectured that that was the entire abelianization, which I believed for a long time, but it turns out that that was wrong.

Rank 2: (Conant-Kassabov-Vogtmann)

$$D^+_{ab}[2] \cong \bigoplus_{k>\ell\geq 0} [2k, 2\ell]_{SP} \otimes \mathcal{S}_{2k-2\ell+2} \oplus$$
$$\bigoplus_{k>\ell\geq 0} [2k+1, 2\ell+1]_{SP} \otimes \mathcal{M}_{2k-2\ell+2}$$

where $\mathcal{S}_w$ is the space of cusp forms of weight w and $\mathcal{M}_w$ is the full space of modular forms of weight w, and $[\lambda]_{SP}$ is a symplectic representation, corresponding to partition λ.

So in rank 2 things are beginning to look more complicated. We're getting all these symplectic representations, with multiplicities given by these modular form spaces. Note that rank 2 is completely done and understood!

We remark that the modular forms here ultimately come from the cohomology of $GL_2(\mathbb{Z})$ with coefficients. In fact, I should mention that the SP-decomposition above is isomorphic to the GL decomposition of $H^1(GL_2(\mathbb{Z}); S(V \otimes k^2))$. Here $GL_2(\mathbb{Z})$ actually acts on the k^2 in the coefficients, and the S here refers to the full symmetric algebra. The different powers will give different weights of modular forms.

Now we move on to rank 3. I will explain the results there and then go back to give the main construction.

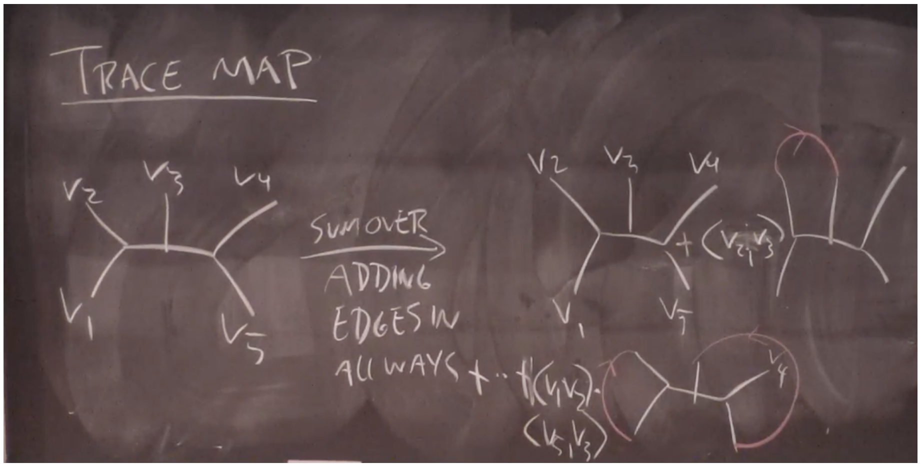
Rank 3:

Theorem 2. *The representation $[a, b, c]_{SP}$ has multiplicity at least $s_{a-b+2} + s_{b-c+2} + \delta_{a,b,c} + \epsilon_{a,b,c}$ in $D^+_{ab}[3]$, where s_w is the dimension of spaces of cusp forms of weight w, $\epsilon_{a,b,c} = 1$ if $a > b > c$ are all even, and 0 otherwise, and $\delta_{a,b,c} = s_{a-b+2}$ if $a - b = b - c$ and 0 otherwise.*

So in rank 3 we're getting a lot of stuff as well. It's the same basic form as the rank 2 case, symplectic representations with multiplicities given by modular forms, but a little more complicated.

Now let me explain where this comes from. It comes from a graphically defined trace map, which is a generalization of the divergence mentioned in Alekseev's talk.

Trace map: Take a tree with different labels and sum over adding external edges in all possible ways.

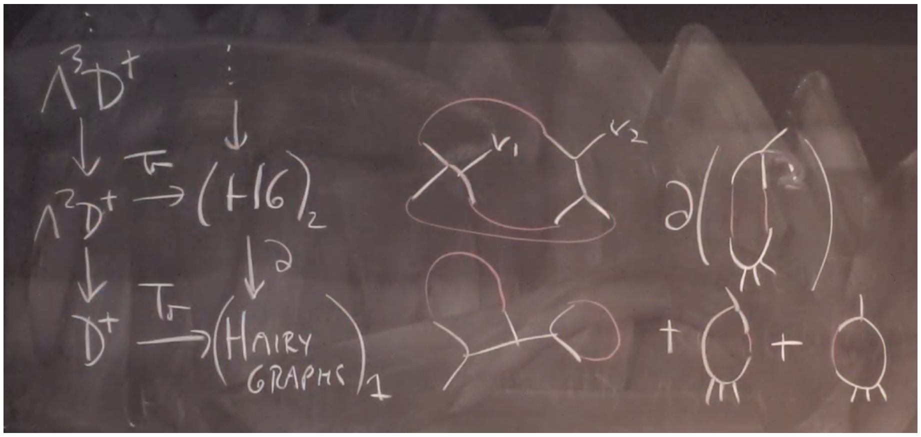


Give the new edge an arbitrary direction,and put a coefficient which is the contraction of the two labels.

Now I want to interpret these graphs with extra edges. I can think of them as being part of a graphical chain complex, which we call hairy graphs. The target of our trace is hairy graphs. Indeed, we have a chain complex:

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \bigwedge^2 D^+ & \xrightarrow{Tr} & (\text{hairy graphs})_2 \\ \downarrow & & \downarrow \partial \\ D^+ & \xrightarrow{Tr} & (\text{hairy graphs})_1 \end{array}$$

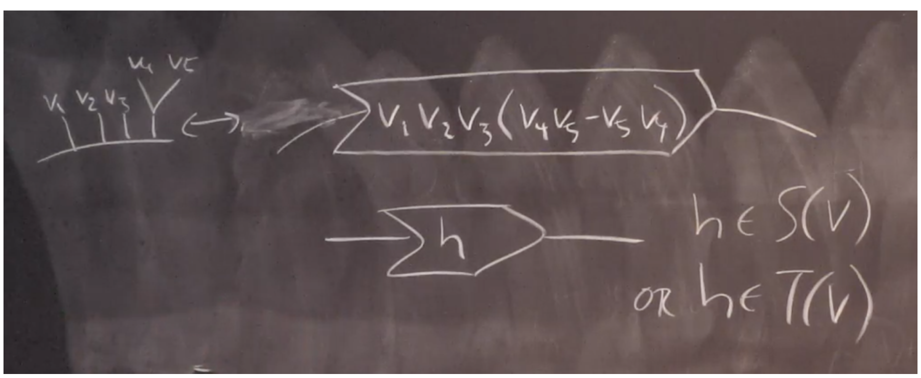
If I'm interested in studying the abelianization, that's the first homology of the Lie algebra D^+, and I have the Chevalley-Eilenberg complex on the left which will compute this. On the right, in degree 1, I have trees with several edges on the outside. In degree 2, hairy graphs are two trees connected up by edges. There's a boundary operator on hairy graphs. You sum over joining trees together along the extra edges.



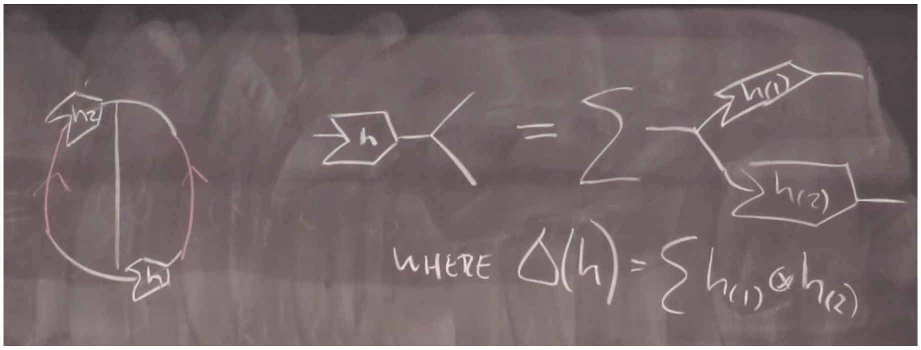
There are some signs and things that I'm ignoring. They are important but also kind of technical. In the above picture, I have the chain complexes on the left, examples of hairy graphs in degrees 1 and 2 in the middle, and an example of a boundary operator on the right.

The theorem is that the trace is a chain map: the graph boundary operator corresponds exactly to taking brackets, and it is injective on homology. So, if we care about the abelianization of D^+, we should care about the first homology of the hairy graph complex.

There's even more we can do at this point, and this will allow me to advertise a little bit more my work with Martin. This is where we get the connection to $Out(F_r)$. If we look at the hairs, we can think of these as elements either of the symmetric algebra or the free associative algebra. In the picture below, I've given an example of this. A string of hairs gives us an element h sitting on an edge where h can be thought of as lying either in $S(V)$ or $T(V)$. (Obviously one carries more information than the other.)



Now I can think of hairy graphs as (non-hairy) graphs with edge labels in a Hopf algebra. (See the left of the picture below.) In fact the IHX relation corresponds to pushing labels through a trivalent vertex via the coproduct $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, as on the right below.



In Karen's talk, we saw that if there are no labels, this corresponds to the cohomology of $Out(F_n)$, with trivial coefficients. This is because the homology of $Out(F_n)$ is a moduli space of graphs (the quotient of Outer Space by the group action). These graphs have no hairs.

This general graph homology construction works for H a cocommutative Hopf algebra H. You also need that the antipode squared is the identity, but this is automatic for the cococommutative case. Now

$$\text{Homology of } H\text{-labeled graphs} \cong \bigoplus_{n\geq 2} H^*(Out(F_n); \overline{H^{\otimes n}})$$

Part of what we did was to define an action of $Aut(F_n)$ on the nth tensor power of a cocommutative Hopf algebra. $\overline{H^{\otimes n}}$ is an appropriate quotient on which inner automorphisms act trivially, so that $Out(F_n)$ acts.

The trace map defines an injective map

$$Tr : D^+_{ab} \hookrightarrow \bigoplus_{n\geq 2} H^{2n-3}(Out(F_n); S(V)^{\otimes n}) \oplus \text{Morita rank 1 part}$$

Rank 1 is a little different since $Out(F_1)$ doesn't correspond to a moduli space of rank 1 graphs.

In fact, the same map induces a map on the Johnson cokernel

$$Tr : C \to \bigoplus_{n\geq 2} H^{2n-3}(Out(F_n); \overline{T(V)^{\otimes n}}) \oplus \text{rank 1 part}$$

and this target is even bigger than that of the abelianization.

Notice that the rank I alluded to earlier is the n in these statements.

Note $S(V)^{\otimes n} \cong S(V \otimes k^n)$ and it turns out that the $Out(F_n)$ action on this factors through the $GL_n(\mathbb{Z})$ action on k^n. On the other hand, the action on $\overline{T(V)^{\otimes n}}$ does not factor through a $GL_n(\mathbb{Z})$ action in general, and even for $n = 2$, when $Out(F_2) = GL_2(\mathbb{Z})$, the action on $\overline{T(V)^{\otimes 2}}$ does not extend to a $GL_2(k)$ action.

We also know, that in both of the above cases, Tr hits every $[\lambda]_{SP}$ inside each $[\lambda]_{GL}$. So for the abelianization we have a complete characterization. For the more general Johnson cokernel case, we understand the image, but the map is not injective. Moreover $H^{2n-3}(Out(F_n); \overline{H^{\otimes n}})$ is not understood at all for $n > 3$. For $H = S(V)$, we used known facts about the homology of $GL_2(\mathbb{Z})$ with coefficients to get a full answer. For $n = 3$, I did a partial computation, but for $n > 3$, it's completely unknown. It's even worse for $H = T(V)$. Martin and I did a bunch of computations for rank 2 and we got stuff that generalizes what you get in the abelianization, and in rank 3 I did some more computations. It's a very difficult question to compute these cohomologies.

2017.03.30 JIM CONANT

The LIE Lie algebra

Joint with various combinations of Karen Vogtmann & Martin Kassabov

$\mathrm{Der}^{\theta}\mathbb{L}(H)$ = "LIE" Lie algebra

H = symplectic vector space

$a_1, \dots, b_g$ symplectic basis

$$\theta = \sum_{j=1}^{g} [a_j, b_j] \in \mathbb{L}(H)$$

Have

$$(\mathrm{Gr}^{\bullet}_{\mathrm{Johnson}}\, T_{g,\vec{1}}) \otimes \mathbb{Q} \hookrightarrow \mathrm{Der}^{\theta}_{<0}\, \mathbb{L}(H)$$

Image generated by

$$(\mathrm{Gr}^{1}\, T_{g,\vec{1}}) \otimes \mathbb{Q} \cong \Lambda^3 H$$

<u>GOAL:</u> Understand $D^{+}_{ab} = H_1(\mathrm{Der}^{\theta}\,\mathbb{L}(H))$

This is very complicated.

D^{+}_{ab} is graded by "rank".

Rank 1 part (Morita):

$$\bigoplus_{n\geq 1} S^{2n+1} H$$

Rank 2 (C-K-V)

$$\bigoplus_{k\geq l\geq 0} V_{[2k,2l]} \otimes S_{2k-2l+2}$$

"cusp forms" of wt $2k-2l+2$

$$\oplus \bigoplus_{k\geq l\geq 0} V_{[2k+1,2l+2]} \otimes M_{2k-2l+2}$$

$Sp(H)$ module

"modular forms"

$$= H^1(GL_2(\mathbb{Z}); Sym(H \otimes \mathbb{Q}^2))$$

$Sp(H)$ GL_2

Rank 3:

$V_{[a,b,c]}$ has multiplicity at least

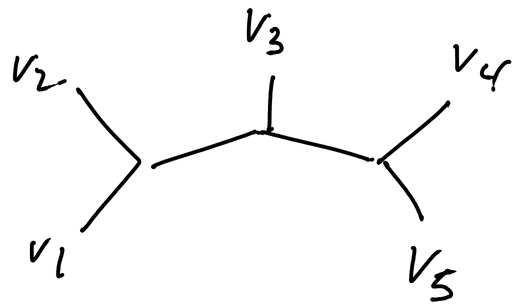
$$s_{a-b+2} + s_{b-c+2} + \delta_{a,b,c} + \varepsilon_{a,b,c}$$

$$s_w := \dim S_w$$

$\varepsilon_{a,b,c} = 1$ if a, b, c even, $a > b > c$

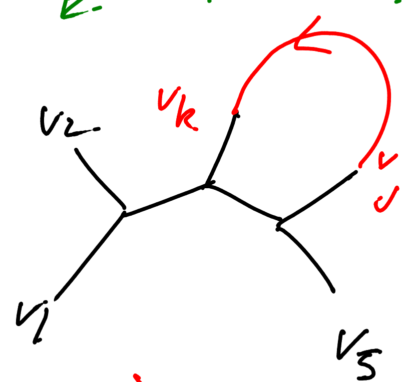
$\delta_{a,b,c} = s_{a-b+2}$ if $a-b = b-c$

Trace map :

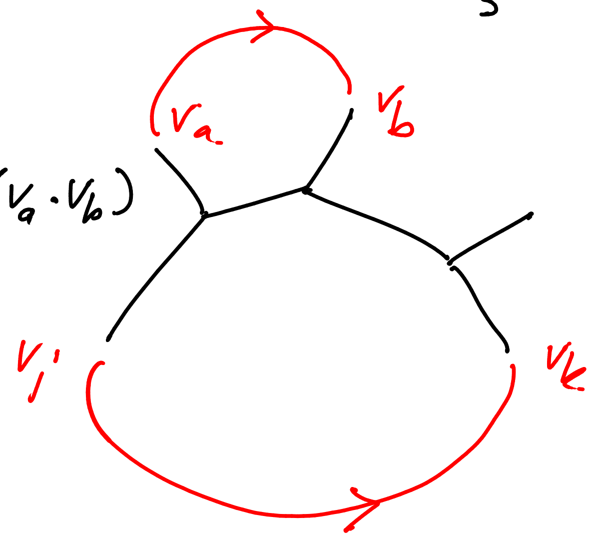


$\longmapsto \quad \sum_{\substack{\text{single}\\ \text{edge}}} (v_j \cdot v_k)$

indep of choice of orient.



$+ \quad \sum_{2 \text{ edges}} (v_j \cdot v_k)(v_a \cdot v_b)$



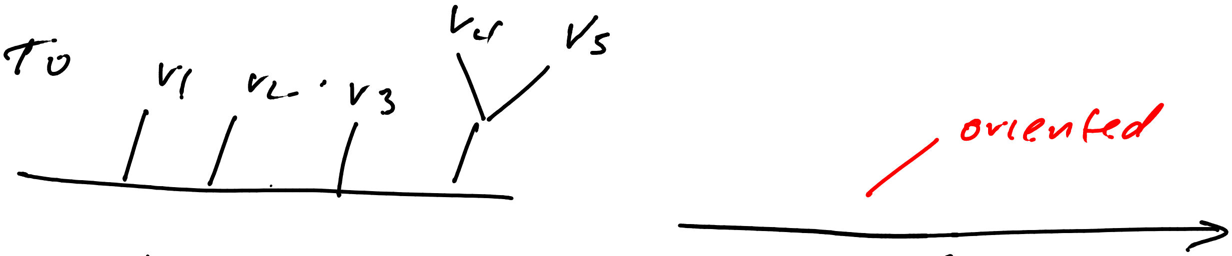
$+ \quad \ldots$

$\rightsquigarrow$ Hairy graph homology $\quad D^+ = \mathrm{Der}^0_{<0}$

$$\begin{array}{ccccc}
\Lambda^3 D^+ & \longrightarrow & \Lambda^2 D^+ & \longrightarrow & D^+ \\
\downarrow \mathrm{Tr} & & \downarrow \mathrm{Tr} & & \downarrow \mathrm{Tr} \\
(HG)_3 & \longrightarrow & (HG)_2 & \longrightarrow & (HG)_1
\end{array}$$

THM: Tr is a chain map.

To

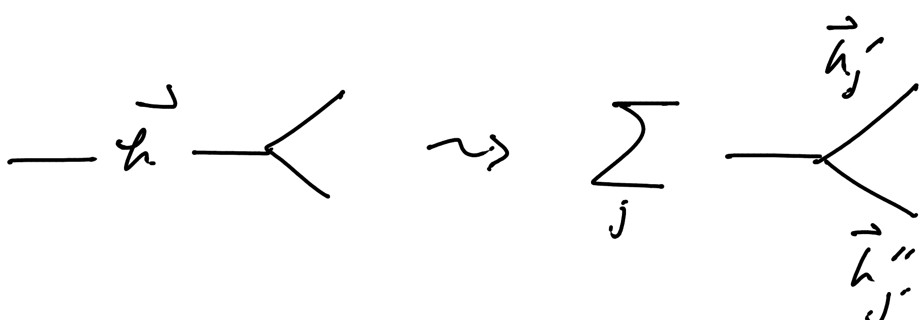


associate $\longrightarrow$ $v_1 v_2 v_3 (v_4 v_5 - v_5 v_4)$

can interpret { in $S(H)$ ← - - - in $T(H)$

These are Hopf algebras.

Think of hairy graphs as graphs with edges labelled by elets of a Hopf algebra. Then see hairy graphs as a Hopf alg.



where $\Delta h = \sum h_j' \otimes h_j''$

No labels: computes $H^\circ(Out F_n; \mathbb{Q})$.

This construction works for any co-comm Hopf algebra A. (Need $(\text{antipode})^2 = id$)

$$\text{Homology} \cong \bigoplus_{n\geq 2} H^{\bullet}(Out\, F_n ; \overline{A^{\otimes n}})$$

$Aut\, F_n$ acts on $A^{\otimes n}$

$Out\, F_n$ acts on quotient $\overline{A^{\otimes n}}$

$Out\, F_n \longrightarrow GL_n$

Trace gives injective homom $Sym(H \otimes k^n) \cong$

$$Tr : D^{+}_{ab} \hookrightarrow \bigoplus_{n\geq 2} H^{2n-3}(Out\, F_n ; Sym(H)^{\otimes n})$$

$\uparrow$ $Sp(H)$ mod

GL_n mod.

$\oplus$ Rank 1 part (Morita trace)

Remarks:

① image: for each partion λ.

get $V_{\lambda_{Sp}}$ in $V_{\lambda_{GL}}$

② In fact, Tr induces a map

$$\underline{k} \hookrightarrow \text{Johnson cokernel} \longrightarrow \bigoplus_{n\geq 2} H^{2n-3}(Out(F_n), \overline{T(V)^{\otimes n}})$$

$\uparrow$ not known to be inj