Motivic Euler numbers and an arithmetic count of the lines on a cubic surface

MSRI Periods
3/30/17

Thank you!

with Jesse Kass

$\deg : [S^n, S^n] \longrightarrow \mathbb{Z}$

$f : S^n \longrightarrow S^n$ smooth, $\quad y \in S^n$

$$\deg f = \sum_{x \in f^{-1}(y)} \deg_x f$$

J = Jacobian determinant
$= \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1}^{n}$

$$\deg_x f = \begin{cases} +1 \iff J_x > 0 \iff f \text{ preserves orientation} \\ -1 \iff J_x < 0 \iff f \text{ reverses orientation} \end{cases}$$

Q: What is $\deg_x f$ when $J_x = 0$?

A: Eisenbud-Levine / Khimshiashvili signature formula

$f : \mathbb{R}^n \to \mathbb{R}^n$ analytic $\quad f = (f_1, \ldots, f_n) \quad f_i \in \mathbb{R}[[x_1, \ldots, x_r]]$

f has an isolated zero at 0.

$Q := \mathbb{R}[[x_1, \ldots, x_n]] / \langle f_1, \ldots, f_n \rangle$ finite dimensional complete intersection

Choose $\eta : Q \to \mathbb{R}$ linear with $\eta(J) = \dim_{\mathbb{R}} Q$

Define $\omega^{EKL} : Q \times Q \to \mathbb{R}$ by $\omega^{EKL}(a,b) = \eta(ab)$

non-degenerate, symmetric bilinear form

isomorphism class ω^{EKL} does not depend on choice of η

Ex: $f(x) = x^2$

$$Q = \frac{\mathbb{R}[x]}{\langle x^2 \rangle} \qquad J = 2x \qquad \omega^{EKL} = \begin{array}{c|cc} & 1 & x \\ \hline 1 & ?=0 & 1 \\ x & 1 & 0 \end{array} \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = h$$

basis $= \{1, x\}$ $\qquad \eta(2x) = 2$

dim $= 2$

Signature formula (EKL '77) Signature $\omega^{EKL} = \deg_0 f$

Palamodov: rank $\omega^{EKL} = \deg_0 f \otimes \mathbb{C}$

Polischuk-Vaintrob: ω^{EKL} is residue pairing on $HH_*(mf(Q,f))$

Question (Eisenbud '78) over any field k, $\omega^{EKL} \in GW(k)$ is defined: replace $\mathbb{R}$ by k above. When char $k \mid \dim Q$ replace J by a certain distinguished socle element.

View ω^{EKL} as degree.

"does this idea of degree have some other interpretation or usefulness, for example in coh thy?"

Thm (Kass–W.) $\omega^{EKL} = \deg_0^{\mathbb{A}^1} f$

where $\deg_0^{\mathbb{A}^1}$ is the local degree in $\mathbb{A}^1$-homotopy thy.

Morel–Voevodsky $\mathbb{A}^1$-homotopy thy '98:

Simplicial model category or ∞-category Spc

tool: Bousfield localization, forces morphisms to be equivalences

$$\mathrm{Sm}/k \xrightarrow{\text{Yoneda}} \mathrm{Functors}(\mathrm{Sm}_k, \mathrm{sSet}) \xrightarrow[L_{Nis}]{} \mathrm{Sh} \xrightarrow[L_{\mathbb{A}^1}]{} \mathrm{Spc}$$

simplicial set or top

Smooth schemes$/k$

$U \to X$ cover becomes equivalence

$X \times \mathbb{A}^1 \to X$ becomes equivalence

Morel: $\deg : [(\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}]_{\mathbb{A}^1} \longrightarrow GW(k)$

S^n

$GW(k)$ = Grothendieck–Witt ring

= group completion of semi-ring $\oplus, \otimes$ of non-degenerate symmetric bilinear forms

generators: $\langle a \rangle$ $\quad a \in k^*/(k^*)^2 \quad (x,y) \mapsto axy$

relation in char $\neq 2$: $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle = h$

Ex: $GW(\mathbb{R}) \xrightarrow[\text{rank} \times \text{signature}]{\cong} \mathbb{Z} \times \mathbb{Z}$, $\quad GW(\mathbb{F}_q) \xrightarrow[\text{rank} \times \text{discriminant}]{\cong} \mathbb{Z} \times \mathbb{F}_q^*/(\mathbb{F}_q^*)^2$

Heuristic: $\mathbb{R} + \mathbb{C} \rightsquigarrow \mathbb{A}^1$-homotopy theory (Morel-Voevodsky)

Applications to Enumerative Geom:

1) $\mathbb{A}^1$-Milnor number: singularity $p \in \{f=0\}$ bifurcates into nodes

node means $\cong x_1^2 + \dots + x_n^2 = 0$ over k^s

$$\# \text{ nodes} = \deg_p^{top} \operatorname{grad} f \qquad \text{(Milnor)}$$

over k? $\quad$ type $(x_1^2 + a x_2^2 = 0) := \langle a \rangle \quad$ char $k \neq 2$

$$\sum_{\text{nodes}} \text{type} = \deg_p^{\mathbb{A}^1} \operatorname{grad} f \qquad \text{(Kass-W)}$$

ex: $\quad f(x,y) = x^3 + y^2 = 0 \quad \operatorname{grad} f = (3x^2, 2y)$

$\deg \operatorname{grad} f = \deg x^2 = \langle 1 \rangle + \langle -1 \rangle$

$\Rightarrow$ bifurcates into 2 nodes

over $\mathbb{F}_5$: $\langle 1 \rangle = \langle -1 \rangle \Rightarrow$ can't bifurcate into a split & non-split

$\mathbb{F}_7$: $\langle 1 \rangle \neq \langle -1 \rangle \Rightarrow$ —— " —— 2 split or 2 non-split

2) Lines on a smooth cubic surface: k field, char $k \neq 2$

$S = \{f = 0\} \subseteq \mathbb{P}^3_k$ smooth cubic surface over $\mathbb{C}$ (Salmon + Cayley 1849) S contains 27 lines

over $\mathbb{R}$ # real lines is 3, 7, 15, 27

(Okonek + Teleman '14
Finashin + Kharlamov '15) A signed count is always 3

$$\underset{\text{lines}}{\#\text{ hyperbolic}} - \underset{\text{lines}}{\#\text{ elliptic}} = 3$$

<u>For proof</u>: $V \longrightarrow X$ rank r vector bundle
on dim r mfld
relatively oriented

Euler number $e(V)$: choose a section $s: X \to V$

$$e(V) = \sum_{\text{zeros } p \text{ of } s} \deg_p s$$

<u>Proof sketch</u>: $X = Gr(1,3)$ parametrises 2 dim'l $W \subset R^{\oplus 4}$
equivalently lines in $\mathbb{P}^3$

$$V = \mathrm{Sym}^3 W^* \longrightarrow X$$

$$V|_{[W]} = \mathrm{Sym}^3 W^*$$

f determines a section $s_f : X \to V$ $\quad s_f([W]) = f|_W$

S contains a line $\ell = [W] \Leftrightarrow s_f([W]) = 0$

$$\Rightarrow \quad e(\mathrm{Sym}^3 W^*) = \sum_{\text{lines } \ell \subseteq S} \deg_\ell s_f$$

over $\mathbb{C}$: $\quad \deg_\ell s_f = 1$

over $\mathbb{R}$: $\quad \deg_\ell s_f = \begin{cases} +1 & \ell \text{ hyperbolic line} \\ -1 & \ell \text{ elliptic line} \end{cases}$

Compute $e(\mathrm{Sym}^3 W^*)$ using splitting principle, $H^*(Gr)$, ...

Alternate description of $\deg_\ell s_f$:

$$\underset{\substack{\text{line} \\ \text{defined over} \\ k(\ell)}}{\ell} \subseteq S \subseteq \mathbb{P}^3_{k(\ell)}$$

$\pi : \mathbb{P}^3_{k(\ell)} - \ell \longrightarrow \mathbb{P}^1 \quad$ projection away from ℓ

ex: $\quad \ell = [0,0,z,w] \quad [x,y,z,w] \mapsto [x,y]$

$\pi|_S : S \longrightarrow \mathbb{P}^1 \qquad \pi^{-1}(pt) = S \cap \text{plane}$

$= \text{line} \cup \text{conic}$

$\pi|_\ell : \ell \to \mathbb{P}^1$ degree 2

$\underset{\text{char } k \neq 2}{\Rightarrow}$ $\exists!$ involution

$$\begin{tikzcd} \ell \arrow[rr, "i"] \arrow[dr] & & \ell \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

over $k(\ell)$

$$\deg_\ell^k s_f = \deg^{top}(\ell(K) \to \ell(K)) \quad K = \mathbb{R} \text{ or } \mathbb{C}$$

$\underset{\text{separable}}{k \subseteq k(\ell)}$ $\quad Tr_{k(\ell)/k} : GW(k(\ell)) \to GW(k)$

$$[A \times A \to k(\ell)] \mapsto [A \times A \to k(\ell) \xrightarrow{Tr} k]$$

<u>Def</u>: $\text{type}(\ell) := Tr_{k(\ell)/k} \deg^{\mathbb{A}^1} i$

<u>Thm</u> (Kass. W) Let k be a field of char $\neq 2$, and let S be a smooth cubic surface over k.

$$\sum_{\text{lines } \ell \subseteq S} \text{type}(\ell) = 15 + 12\langle -1 \rangle$$

<u>Rmk</u>: Applying rank recovers result over $\mathbb{C}$

Applying Signature —— " —— $\mathbb{R}$

(lines over $\mathbb{C}$ contribute h)

About proof: $e(V) \in GW(k)$ $\quad$ V relatively oriented

s section with isolated zeros

$$e(V) = \sum_{\substack{p \text{ such} \\ \text{that} \\ s(p)=0}} \deg_p s$$

Show $\deg_\ell s_f = \text{type}(\ell)$ for $V = \mathrm{Sym}^3 \mathcal{W}^* \to X = Gr(1,3)$

independence of section, when space of sections with isolated zeros is naïvely $\mathbb{A}^1$-connected: s, s' sections with $ts+(1-t)s'$ has isolated zeros

$$\begin{tikzcd}
V \arrow[d] & \\
X \times \mathbb{A}^1 \arrow[d] & \supseteq Z = \{ts+(1-t)s' = 0\} \arrow[dl, "\pi"] \\
\mathbb{A}^1 &
\end{tikzcd}$$

Relative orientation $\mathcal{L} \to Z$ line bundle

Family of bilinear forms $\pi_* \mathcal{L}|_Z \otimes \pi_* \mathcal{L}|_Z \to \mathcal{O}_{\mathbb{A}^1}$

$$(a,b) \longmapsto \eta(ab)$$

η defined in families: Scheja-Storch $\square$

Alternate approach to Euler #: $\widetilde{CH}$ Barge-Morel, Fasel

Push-forward Euler class Morel, Asok-Fasel, M. Levine

different from M. Levine from W^{EKL}

More applications of $\mathbb{A}^1$-homotopy to enumerative geometry:

Marc Hoyois "A Quadratic Refinement to the Grothendieck-Lefschetz-Verdier Trace Formula" '14

Marc Levine "Toward an Enumerative Geometry with Quadratic Forms" '17

2017.03.30 Kirsten Wickelgren

Joint with Jesse Kass

$$\deg : [S^n, S^n] \to \mathbb{Z}$$

$$f : S^n \to S^n$$

$$\deg f = \sum_{x \in f^{-1}(y)} \deg_x f \qquad \text{some } y \in S^n$$

$$\deg_x f = \begin{cases} +1 & \text{if } J(x) > 0 \\ -1 & J(x) < 0 \end{cases}$$

jacobian of f
$= \det Df(x)$

Q: what is $\deg_x f$ if $J(x) = 0$.

Eisenbud–Levine / Khimshiashvili Signature formula

$f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$ isolated zero

$f = (f_1, \dots, f_n)$ analytic

$f_i \in \mathbb{R}[[x_1, \dots, x_n]]$

$$Q(f) = \mathbb{R}[[x_1, \dots, x_n]] / (f_1, \dots, f_n)$$

finite dimensional complete intersect$^{\underline{n}}$.

Choose

$$\eta : Q(f) \to \mathbb{R}$$

s.t. $\eta(J) = \dim Q(f)$

Define

$$\omega^{EKL} : Q(f)^{\otimes 2} \longrightarrow \mathbb{R}$$
$$a \otimes b \longmapsto \eta(ab)$$

Isom class of ω^{EKL} does not depend on η. It is non-degen & symmetric.

Signature formula: (EL/K 77)

$$\deg_0 f = \text{signature of } \omega^{EKL}$$

Palamodov:

$$\deg_0 f \otimes \mathbb{C} = \operatorname{rk} \omega^{EKL}$$

<u>Question</u> (Eisenbud, 78) k field. Define ω^{EKL} by replacing $\mathbb{R}$ by k

Thm (Kass–Wickelgren)

$$\omega^{EKL} = \deg_0^{\mathbb{A}^1}(f)$$

in $GW(k)$ is the local degree in $\mathbb{A}^1$-

Grothendieck–Witt gp

homotopy theory.

Morel–Voevodsky: $\mathbb{A}^1$ homotopy theory '98

$$\text{schemes}/k \xrightarrow{\text{Yoneda}} \text{Functor}(Sm_k, \underline{\text{Sets}}^{\Delta})$$

Smooth Schemes / k

$$\xrightarrow{L_{Nis}} Sh$$

$$\xrightarrow[L_{\mathbb{A}^1}]{} Spc$$

Homotopy theory: simplicial model cat, or ∞-category

Tool: Bousfield localization.

S^n has analogue in Spc: viz $(\mathbb{P}^1)^{\wedge n}$

Morel: $\deg^{\mathbb{A}^1}: [(\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}] \to GW(k)$

$GW(k)$ is the group completion of the semi-ring (under $\oplus, \otimes$) on non-degen symmetric bilinear forms.

It has generators $\langle a \rangle$, $a \in k^\times/(k^\times)^2$:

$$\langle a \rangle (x,y) = a\,xy$$

Relation: $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$ when char $k \neq 2$.

Examples:

① $GW(\mathbb{C}) \xrightarrow{\cong} \mathbb{Z}$ rank

② $GW(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}$ (rank, signature)

③ $GW(\mathbb{F}_q) \longrightarrow \mathbb{Z} \oplus \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ (rank, discriminant)

Heuristic:

$\mathbb{R} + \mathbb{C} \rightsquigarrow \mathbb{A}^1$ homotopy theory + realization functors

<u>Applications</u> to enumerative geometry

① $\mathbb{A}^1$ – Milnor number: a singularity $p \in f^{-1}(0)$ into nodes

$\swarrow$ sep closure

$(x_1^2 + \cdots + x_n^2 = 0$ over $k^s)$

over $\mathbb{C}$: # nodes $= \deg_p f$

Over k ?

For $x_1^2 + a x_2^2 = 0$, $\deg_0 f = \langle a \rangle$

(Kass–W) $\sum_{P \text{ a node}} \text{type}(p) = \deg_p \cdot \text{grad} f$

when char $k \neq 2$.

<u>Ex:</u> $f = \frac{1}{3} x^3 + y^2/2$ $\quad P = 0$, char $\neq 2, 3$

$\text{grad}(f) = (x^2, y)$

$\deg_0(f) = \deg_0(x \mapsto x^2) \deg_0(y \mapsto y)$

$\deg_0(x^2)$:

$Q(f) = k[\![x]\!]/\langle x^2 \rangle = k + kx.$

$J = 2x \qquad \eta : k[x]/(x^2) \to k$

$\eta(2x) = 2$

$\eta(0) = 0$

$w^{EKL} = x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \langle 1 \rangle + \langle -1 \rangle$

Lines on a cubic surface

$S = \{ f = 0 \}$ $\quad f$ homog, deg 3

$f \in k[x, y, w, z]$

S smooth in $\mathbb{P}^3$

$k = \mathbb{C}$ (Salmon + Cayley, 1849)

$\#$ lines in $S = 27$

$k = \mathbb{R}$ $\begin{pmatrix} \text{Okonek + Teleman '14} \\ \text{Finashin + Khalamov '15} \end{pmatrix}$

$\#$ hyperbolic lines $-$ $\#$ elliptic lines $= 3$

$\#$ real lines can be $3, 7, 15, 17$

For proof: $V \to X$ rk r vector bdle. on r-manifold, oriented

$$e(V) = \sum_{p \text{ zero of } s} \deg_p s$$

$$\begin{tikzcd} V \arrow[d] \\ X \arrow[u, bend right, "s"'] \end{tikzcd}$$

s sect. with isol zeros

pf sketch ($\mathbb{R}$ or $\mathbb{C}$) $\quad X = G(1,3)$

$= \{ l \subseteq \mathbb{P}^3 \}$

$= \{ W \subseteq k^4 \ \dim 2 \}$

$V = \mathrm{Sym}^3 W^* \longrightarrow X \qquad \begin{cases} W \to G(1,3) \\ \text{tautological bdle} \end{cases}$

f gives section s_f

Thm follows from $\deg s_f = 1 \quad k = \mathbb{C}$

$= \left. \begin{cases} 1 & l \text{ hyp} \\ -1 & l \text{ ell} \end{cases} \right\} k = \mathbb{R}$