# Galois theory for motivic cyclotomic multiple zeta values (CMZV)

`Algebra of periods`: $\mathcal{P}$ `Period`: $\int_\gamma \omega$, algebraic function, algebraic domain.

$\mathcal{P}$ contains $\pi$, zeta values, MZV, $\log(\alpha)$, $\alpha \in \overline{\mathbb{Q}}$ and many of interesting constants. Should be extended by $\frac{1}{\pi}$ as Annette explained.

## 1 MZV$_{\mu_N}$ at roots of unity

`Definition` (for $N = 1$, usual MZV)

$$\mathtt{MZV}_{\mu_N}: \quad \zeta\begin{pmatrix} n_1, \dots, n_p \\ \epsilon_1, \dots, \epsilon_p \end{pmatrix} := \sum_{0<k_1<k_2\cdots<k_p} \frac{\epsilon_1^{k_1}\cdots\epsilon_p^{k_p}}{k_1^{n_1}\cdots k_p^{n_p}} \qquad \text{where } \begin{cases} n_i \in \mathbb{N}^*, \epsilon_i \in \mu_N \\ (n_p, \epsilon_p) \neq (1,1) \end{cases} \tag{1}$$

They have writing in **terms of the following** iterated integrals (hence periods):

$$= (-1)^p I(0; \eta_1 0^{n_1-1} \cdots \eta_p 0^{n_p-1}; 1)$$

$$\texttt{where} \quad \boldsymbol{I(0; a_1, \dots, a_n; 1)} := \int_{0<t_1<\cdots<t_n<1} \omega_{a_1}(t_1)\cdots\omega_{a_n}(t_n) \quad , \text{ and } \quad \omega_a = \frac{dt}{t-a}$$

with $\eta_i = (\epsilon_i \cdots \epsilon_p)^{-1}$.
In these notations, the weight is $w = \sum n_i$, the depth is $p$.

`Examples`: *In depth 1*, $\rightsquigarrow$ values classical polylog at roots unity:

$$\zeta\begin{pmatrix} n \\ \epsilon \end{pmatrix} = (-1)^n I(0; \epsilon^{-1} 0^{n-1}; 1) = Li_n(\epsilon)$$

*In depth 1, weight 1:*[I]

$$\zeta\begin{pmatrix} 1 \\ \epsilon \end{pmatrix} = -log(1-\epsilon^{-1}) \rightsquigarrow \texttt{cyclotomic units modulo torsion}$$

**Relations** :
In depth 1, we conj know all relations (easy exercice to check them). They are coming from:

- `Conjugation` relation, here modulo $(2i\pi)$:

$$\zeta\begin{pmatrix} n \\ \epsilon \end{pmatrix} \cong (-1)^{n-1} \zeta\begin{pmatrix} n \\ \overline{\epsilon} \end{pmatrix} \qquad \xi_n^a - 1 = -\xi_n^a(\xi_n^{-a} - 1)$$

[I]Related to cyclotomic units modulo torsion. Cf. Algorithm to build a basis of cyclotomic units modulo torsion for any N for instance in a recent article of Conrad.

- `Cyclotomic`[I] relation, for each $d$ dividing $n$, $\epsilon \in \mu_{\frac{N}{d}}$:[II]

$$\zeta\binom{n}{\epsilon} = (d)^{n-1} \sum_{\eta^d=\epsilon} \zeta\binom{n}{\eta}$$

  `Nota Bene`: This distribution easily generalises to higher depth.

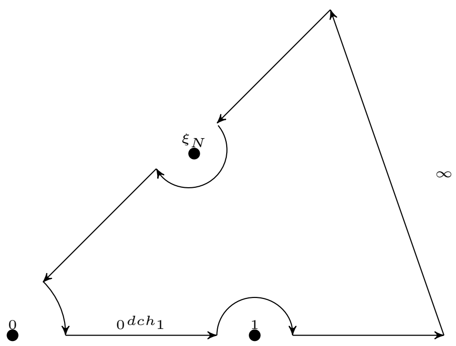
What about higher depth? These numbers have a rich algebraic structure [III], but let's mention the most important:

- `Shuffle` relation coming from multiplication of iterated integrals (II).
  `Ex`: $\zeta(2)\zeta(3) = 3\zeta(2,3) + 6\zeta(1,4) + \zeta(3,2)$

- `Stuffle` coming from multiplication of series.
  $\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(5) + \zeta(3,2)$.
  These 4 previous relations, once lifted to higher depths, and regularised, are refered to as `standard relations` by Zhao, and for some N, 1,2,3, or power of prime greater than 3, are conjecturally sufficient to generate all relations between MZV at roots of unity, but not for the other N.

- MMZV at roots of unity are linked with the geometry of $X_N = \mathbb{P}^1 \diagdown 0, \infty, \mu_N$.
  `Octogon` (hexagon for $N = 1$) correspond to a contractible path in $X_N$, cf. figure below.
  By Betti de Rham comp isom, this leads to bunch of identities, written in terms of the generating series (cf.Drinfeld associator); note that different segments in these paths are related to the straight path via symmetries (permutation and dihedral symmetry of $X_N$).

- `Pentagon` relation for $N = 1$, in same vein, corresponding to a contractible path in $\mathcal{M}_{0,5} = \{(0,1,x,y,\infty) : x, y \in \mathbb{P}^1 \diagdown \{0,1,\infty\}, x_{\text{¸}} \neq y\}$. It lifts for other roots of unity to the degree $N^2$ covering $\tilde{\mathcal{M}_{0,5}}$ of $\mathcal{M}_{0,5}$. [IV]




`Definition`: $\mathcal{Z}_n^N$ as $\mathbb{Q}$vector space gen by $MZV_N$ of weight $n$.

`Question`: Dimension?

`Fact`: *(comes from motivic theory, as see later)*, there is an upper bound for dim: $d_n^N \leq D_n^N$ where

$$\sum D_n^N t^n = \frac{1}{1-(a_N+1)t+(a_N-b_N)t^2}, N > 2 \quad \frac{1}{1-t+-t^2} N = 2,$$

`where` $a_N = \frac{\phi(N)}{2} + p(N) - 1$, $b_N = \frac{phi(N)}{2}$.
However, this upper bound not reached for $N = p^s$, $p > 5$.

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[I] Also called distribution relation.

[II] Since $\sum_{\eta^d=\epsilon} \eta^k = d\epsilon^{\frac{k}{d}}$ if d divides k, 0 else.

[III] Remind that the only relations between periods should (conj) come from rules of elementary integral calculus: additivity integrand or domain, invertible change variables, Stokes form

[IV] Cf. Hadian.

**Cohomological point of view** *Another way of seeing those periods, as Annette explained is as coeff pairing between relative algebraic de Rham cohomology and relative singular homology.*[I]

A **period** is a coefficient of the pairing between algebraic De Rham cohomology and relative singular homology: $\int_\gamma \omega = \langle [\gamma], comp_{B,dR}([\omega]) \rangle$.

`Setup here`: k a cyclo field.

- $X$ smooth algebraic variety$_{/k}$, $Y$ closed subvariety$_{/k}$.
- $[\omega] \in H^n_{dR}(X,Y)$, $\omega$ closed algebraic $n$-form on X whose restriction on Y is zero.[II]
- $[\gamma] \in H^B_n(X,Y)$,$\gamma$ is a singular $n$ chain on C points of $X$ whose border is in Y. For Betti homology, we have to fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$.
- `Comparison isomorphism` due to Grothendieck, once tensor by $\mathbb{C}$:

$$\mathrm{comp}_{B,dR} : H^\bullet_{dR}(X,Y) \otimes_k \mathbb{C} \underset{\sim}{\longrightarrow} H^\bullet_B(X,Y) \otimes_{\mathbb{Q}} \mathbb{C}$$

`Ex`:

- $X = \mathbb{P}^1 \smallsetminus \{0,\infty\}, Y = \varnothing$: $\gamma_0$ counterclockwise loop around 0:

$$H^B_1(X) = \mathbb{Q}\left[\gamma_0\right] \qquad H^1_{dR}(X) = \mathbb{Q}\left[\frac{dx}{x}\right], \quad \int_{\gamma 0} \frac{dx}{x} = 2i\pi$$

  Period of Lefschetz motive $\mathbb{L} := \mathbb{Q}(-1) = \mathcal{H}^1(X)$, dual to Tate motive $\mathbb{Q}(1)$.

- For $X = \mathbb{P}^1 \smallsetminus \{0,\infty\}, Y = \{1,n\}$:

$$H^B_1(X,Y) = \mathbb{Q}\left[\gamma_0\right] \oplus \mathbb{Q}\left[\delta_{1,n}\right] \qquad H^1_{dR}(X,Y) = \mathbb{Q}\left[dx\right] \oplus \mathbb{Q}\left[\frac{dx}{x}\right]$$

  Hence the period matrix: $\begin{pmatrix} 1 & log(n) \\ 0 & 2i\pi \end{pmatrix}$.

  `Variant` For $X = \mathbb{P}^1 \smallsetminus \{\xi,\infty\}, Y = \{0,1\}$, period matrix: $\begin{pmatrix} 1 & log(1-\xi^{-1}) \\ 0 & 2i\pi \end{pmatrix}$

`Remark`: Often given a period, usually hard to find X and Z such that is coefficient of the associated period matrix. For most interesting periods, poles of diff forms meet integration domain, and has to blow up points to avoid singularities: already for $\zeta(2)$, some work to define it properly like this.

[I] Defined up to choice of basis of Betti homology and de Rham cohomology.
`Nota Bene`:For affine var, all class in de Rham cohomology represented by differential form and this pairing corresponds to integ.

[II] Hypercohomology of the complex of algebraic (rational and holo) differential forms on X (Kahler). Agree with the hypercohomology of the analytic de Rham complex (smooth diff form) for affine variety. For smooth complex varieties isomorphic (Groth) to the usual smooth de Rham cohom (exterior derivative), coefficients in k.

## 2 MMZV$_{\mu_N}$

`Motivic scenery`: k cyclotomic field, [I]

- There is a `tannakian category` $\boldsymbol{\mathcal{MT}(k)}_{\mathbb{Q}}$ of Mixed Tate Motives over $k$ with rational coefficients, ie every object $M \in \mathcal{MT}(k)_{\mathbb{Q}}$ is an iterated extension of Tate motives $\mathbb{Q}(r) = \mathbb{Q}(1)^{\otimes r}, r \in \mathbb{Z}$

- equipped `weight filtration` $W_{-2r}$ indexed by even negative integers[II]. This defines a:

- `fiber functor` $\begin{array}{rl} \omega : & \mathcal{MT}(k) \to \mathrm{Vec}_{\mathbb{Q}} \\ & M \mapsto \oplus \mathrm{Hom}_{\mathcal{MT}(k)}(\mathbb{Q}(r), gr^W_{-2r}(M)) \end{array}$.

- Tannakian category hence a category of (finite dimensional) representations of a `Motivic Galois group` $\boldsymbol{\mathcal{G}^{\mathcal{MT}}}$; a motive can be seen as vector space (realisation) more an action of this Galois group.

$$\mathcal{MT}(k)_{\mathbb{Q}} \cong \mathrm{Rep}_k \mathcal{G} \cong \mathrm{Comod}\ (\mathcal{O}(\mathcal{G})) \cong \mathrm{Comod}\ {}^{gr}\mathcal{A} \tag{2}$$

- `the motivic Galois group :`$\boldsymbol{\mathcal{G}} := \mathrm{Aut}^{\otimes}\omega = \mathbb{G}_m \ltimes \mathcal{U}$ where $\mathcal{G}^{\mathcal{MT}}$ decomposes (since $\omega$ graded for MT, the pro-reductive part is simply $\mathbb{G}_m$), as semidirect product multiplicative group and prounipotent part U.

- `The fundamental Hopf algebra:` $\boldsymbol{\mathcal{A}} := \mathcal{O}(\mathcal{U}) \cong$.
  By Borel results on K theory, we know the extension groups, and the dimension of $\mathcal{A}_n$ as a $\mathbb{Q}$ v.s.[III]

- We can also define a `tannakian subcategory`: $\mathcal{MT}_{\Gamma}$, for $\Gamma$ sub-vector space of $k^* \otimes \mathbb{Q} = Ext^1_{MT(k)}(\mathbb{Q}(0), \mathbb{Q}(1))$ such that $\mathcal{A}_{\Gamma}$ maximal sub-Hopf algebra in $\mathcal{A}_k$ such that $\mathcal{A}_1 \cong \Gamma$.
  Cf. the isomorphism : $log^{\mathfrak{a}}(a) \leftrightarrow a$.[IV]

- HERE, for CMZV, interested in the following categories,

$$\texttt{Category}: \quad \mathcal{MT}(\mathcal{O}) \quad \subsetneq \quad \mathcal{MT}_N \quad \subset \quad \mathcal{MT}(\mathcal{O}[\frac{1}{N}])$$

  The seconde inclusion is an equality iff N has all its prime factors $p$ inert ie generating $(Z/mZ)^*$, for $N = p^{v_p(n)}m$.

$$\texttt{Corresponding } \Gamma: \quad \mathcal{O}^* \subsetneq \begin{array}{c}\text{cyclotomic units} \\ \text{in} \quad \mathcal{O}[\frac{1}{N}]\end{array} \subsetneq (\mathcal{O}[\frac{1}{N}])^* \quad \cdots \text{ modulo torsion!!}$$

  The motivic MZV$_{\mu_N}$are in $\mathcal{MZ}_N \subset \mathcal{MT}_N$.
  The fundamental Hopf algebra (for $N > 2$[V]) has $b_N = \frac{\phi(N)}{2}$ generators in degree $> 1$, $a_N$ in degree 1

$$\mathcal{A} \cong_{n.c} \mathbb{Q}\langle f_i^{(j_i)} \rangle, \quad \sqcup\!\sqcup, \Delta_D, \quad f_i^{(j)} \text{degree i}$$

$$\text{with } a_N respectively: \frac{\phi(N)}{2} - 1 \quad \text{resp.} \quad \frac{\phi(N)}{2} + p_N - 1 \quad \text{resp.} \quad \frac{\phi(N)}{2} + n_N - 1$$

  By Dirichlet theorem for S integers, with $n_M$ the number of prime ideals in O above the primes dividing N, and $p_N$ the number of primes dividing N.

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[I] Hence conjecture annulation Beilinson Soule ok here.

[II] $gr^W_{-2r}M$ direct sum of $\mathbb{Q}(r)$

[III] $\mathrm{Ext}^1_{\mathcal{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(k)_{\mathbb{Q}} \otimes \mathbb{Q} \cong \begin{cases} k^* \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } n = 1. \\ \mathbb{Q}^{r_1 + r_2} & \text{if } n > 1 \text{ odd} \\ \mathbb{Q}^{r_2} & \text{if } n > 1 \text{ even} \end{cases} \quad \mathbb{Q}^{\frac{\phi(n)}{2}} \text{ for } n > 1, N > 2.$

[IV] $Ext^1$ is $\Gamma$ : $\forall \quad 0 \to \mathbb{Q}(n+1) \to E \to \mathbb{Q}(n) \to 0$, E is in $\Gamma$.

[V] For $N = 1, 2$, 1 generators in each odd degree $> 1$, resp $\geq 1$.

**Motivic periods** Let $\mathcal{M}$ our cat of MTM, and look finally at its motivic periods.

- `Algebra of motivic periods`: alg of affine function on bitorsor of tensor preserving isomorphisms of functor:

$$\texttt{Algebra motivic periods :} \quad \boldsymbol{\mathcal{P}^{\mathfrak{m}}_{\mathcal{M}}} := \mathcal{O}(Isom^{\otimes}(\omega, \omega_B))$$

  between fiber functor and Betti realisation functors,[I],once fixed an embedding from k to C.
  *Note the groupoid structure (composition) for isomorphism of functors would lead, by dualising to a comultiplication..*
  `Motivic period` denoted as a triplet, with a motive of this MTM cat, and classes in realisations of M:[II]

$$\texttt{Motivic period :} \quad \boldsymbol{[M, v, \sigma]^{\mathfrak{m}}} \quad M \in \text{ Ind } (\mathcal{M}), \quad v \in \omega(M), \sigma \in \omega_B(M)^{\vee}$$

- Its **period** is obtained by evaluation on the complex point $\text{comp}_{B,dR}$:

$$\texttt{Its period} \quad : per([M, v, \sigma]^{\mathfrak{m}}) := \langle \text{comp}_{B,dR}(v \otimes 1), \sigma \rangle$$

  Here the **period morphism**, conjectured isomorphism (grothendieck):

$$\texttt{period morph :} \quad \text{per} : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}.$$

`Nota Bene`: Following this definition, have a right action motivic Galois group on motivic periods

$$\mathcal{P}^m \times \mathcal{G} \to \mathcal{P}^m$$

$\rightsquigarrow$ (under period conjecture) a *Galois theory of periods.*

`Examples:`

- `Motivic` $2i\pi$ Lefschetz motivic period $\mathbb{L}^{\mathfrak{m}} = [H^1(\mathbb{G}_m), [\frac{dx}{x}], [\gamma_0]] = (2i\pi)^{\mathfrak{m}}$.
- `Motivic log` motivic period of Kummer motive $K_p = H^1(\mathbb{P}^1 \diagdown \{0, \infty\}, \{1, p\})$, in $\mathcal{MT}(\mathbb{Q})$[III]
  $log^{\mathfrak{m}}(p) = [K_p, [\frac{dx}{x}], [\delta_{1,p}]]$
- `Motivic II`: *Motivic period associated to motivic fundamental groupoid.*

$$I^{\mathfrak{m}}(0; w; 1) := \left[\mathcal{O}\left({}_0\Pi_1^{\mathfrak{m}}\left(X_N\right)\right), w, {}_0\, dch_1^B\right]^{\mathfrak{m}}$$

  with ${}_0 dch_1^B$ image of straight path from 0 to 1 in Betti realisation dual.
  Its period is the corresponding II $I(0; w; 1)$.
  Once defined these motivic iterated integrals (MII), can define $\texttt{MMZV}_{\mu_N}$.

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[I] $\omega_{dR}$ in $Vec_k$ but $\omega$, $\omega_B$ in $Vec_{\mathbb{Q}}$. $\omega_{dR} \cong \omega \otimes_{\mathbb{Q}} k$.

[II] It is a function $\boldsymbol{Isom^{\otimes}(\omega, \omega_B)} \to \mathbb{A}^{\mathbf{1}}$, which, on its rational points is: $\boldsymbol{\alpha} \mapsto \boldsymbol{\langle \alpha(v), \sigma \rangle}$.

[III] $\text{Ext}^1_{\mathcal{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \mathbb{Q}^* \otimes_{\mathbb{Z}} \mathbb{Q} \cong \oplus_{p \text{ prime}} \mathbb{Q}$, in Mixed Motive category, should have short exact sequence (implied by long exact sequence in cohomology) $0 \to \mathbb{Q}(1) \to K_p \to \mathbb{Q}(0) \to 0$

**Aparte, about motivic fund gp** :

- `Betti`:
  `fund gpoid` : $\pi_1(X_N, x, y)$ freely generated by $\gamma_0, \cdots, \gamma_N$
  `Prounipotent (Malcev) completion`, affine group scheme, is the **Betti** realisation of $\pi_1^m$:

$$_x\Pi_y^B = \boldsymbol{\Pi^{un}(X_N, x, y)} : R \mapsto \left\{ S \in R\langle\langle e_0, (e_\eta)_{\eta \in \mu_N} \rangle\rangle^\times \mid \begin{array}{l} \Delta S = S \otimes S \\ \epsilon(S) = 1 \end{array} \right\}$$

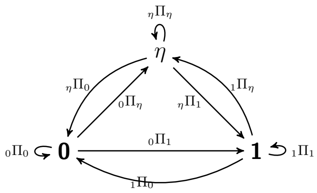
  [I] `Remark`: Using Beilison theorem, can show that the ring of function over this prounipotent completion defines a mixed Tate object in $MT(k)$. Goncharov, in the case of tangential base points (only! *From now on, tangential base points*), can show that it has a good reduction outside $N$hence:

$$\mathcal{O}\left({}_x\Pi_y^{\mathfrak{m}}\right) \in \text{ Ind } \mathcal{MT}\left(\mathcal{O}_N\left[\frac{1}{N}\right]\right) \subset \text{Ind } \mathcal{MT}(k_N). \tag{3}$$

- $\omega(\mathcal{O}({}_0\Pi_1^{\mathfrak{m}})) \cong \mathbb{Q}\left\langle \omega_0, (\omega_\eta)_{\eta \in \mu_N} \right\rangle$ graded Hopf alg with ⧢ product, deconcatenation coprod.

- `Groupoid structure` of $\Pi^{\mathfrak{m}}$ roughly represented below.

- `Dihedral action` $Di_N = Z/2Z \ltimes \mu_N$ acts by permuting tangential base points. Can restrict to the motivic bitorsor of path ${}_0\Pi_1^{\mathfrak{m}}$




[I]The completion of $\pi_1(X_N, x, y)$ is the Hopf algebra of all non commutative formal series with $N+1$ generators, coproduct such that $e_i$ primitive, $\hat{\Pi} = lim_{\leftarrow}\mathbb{Q}[\Pi]/I^n$. Its affine ring of regular functions is the graded Hopf algebra with ⧢ product, deconcatenation coproduct: $\boldsymbol{\mathcal{O}(\Pi^{un}(X_N))} \cong \mathbb{Q}\left\langle e^0, (e^\eta)_{\eta \in \mu_N} \right\rangle$.

**Motivic MZV$\mu_N$ :**

- Let $\boldsymbol{H^N} = \mathcal{P}^{\mathfrak{m},+}_{\mathcal{MZ}_N}$ $\mathbb{Q}$-subvector space generated by MMZV at N roots unity.
  MMZV$_{\mu_N}$ are geometric[I] motivic periods of $\boldsymbol{\mathcal{MZ}_N}$ generated by the motivic fundamental group of $\mathbb{P}^1 \diagdown 0, \infty, \mu_N$.[II]
  Since, $\mathcal{MZ}_N \subset \mathcal{MT}_N \rightsquigarrow$ `Upper bound on dimensions` of $\mathcal{MZ}_N$.[III]
  `Sometimes`: these space are equals: $\pi_1^{\mathfrak{m}}$ generate $\mathcal{MZ}_N$. `Proven for:` $N = 1, 2, 3, 4, 6, 8$, Brown, Deligne.
  `Not equal`: for $N = p^s$ prime $> 3$.

- The period morphism here:

$$\begin{array}{llll} \text{per}: & \mathcal{H}^N & \rightarrow & \mathcal{Z}^N \\ & \zeta^{\mathfrak{m}}(\cdot) & \mapsto & \zeta(\cdot) \end{array}.$$

  Permit us to deduce results from cyclotomic MMZV on complex numbers: a basis for MMZV gives generating family for MZV. Moreover, `conjecturally same`, do not loose any information.

- `Hopf algebra struct` of MMZV$_{\mu_N}$, brought by dualising motivic Galois action on motivic periods:

$$(Goncharov, Brown) \quad \texttt{explicit coaction on MMZV}_{\mu_N} \quad \boldsymbol{\Delta} : \mathcal{H}^N \rightarrow \mathcal{A}^N \otimes_{\mathbb{Q}} \mathcal{H}^N$$

  This coaction has an explicit a combinatorial form, which enables to prove relation up to rational coefficients, but notably linear independence results.
  The weight graded part of the coaction:

$$D_r I^{\mathfrak{m}}(a_0; a_1, \cdots, a_n; a_{n+1}) = \sum_i I^{\mathfrak{a}}(a_i; a_{i+1}, \cdots, a_{i+r}; a_{i+r+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \cdots, a_i, a_{i+r+1}, \cdots a_n; a_{n+1})$$

  It also conveys the Galois descents informations, as we will see below. `Ex`:

$$D_1\left(\zeta^{\mathfrak{m}}\left(\begin{array}{c} 3,3 \\ -1,-1 \end{array}\right)\right) = D_5(\zeta^{\mathfrak{m}}\left(\begin{array}{c} 3,3 \\ -1,-1 \end{array}\right)) = 0$$

[I] Generated by motives with positive weights.
[II] Beware, for $N = 1, 2$, consider the subset $\mathcal{P}^{\mathfrak{m},+}_{\mathcal{M},\mathbb{R}}$ invariant by real Frobenius.
[III] And hence of $\mathcal{Z}_N$.

# 3 Motivic Galois Theory For $\text{MMZV}_{\mu_N}$

**Motivic Galois theory** As seen, there is a right action of motivic Galois group on motivic periods. If `conjecture of periods` hold, for each period $p$, there would be **Galois conjugates**, and a **Galois group** associated to $p$ [I] which permutes them.

`Examples`:

| Period | Conjugates | Galois repr | rank | deg Unip | Hodge polynomial | Galois gp |
|---|---|---|---|---|---|---|
| $(\pi^{\mathfrak{m}})^k$ | $\mathbb{Q}^*(\pi^{\mathfrak{m}})^k$ | $(\lambda_g^k)$ | 1 | 0 | $(uv)$ | $\mathbb{G}_m$ |
| $\log^{\mathfrak{m}}(p)$ | $\mathbb{Q}^*\log^{\mathfrak{m}}(p)+\mathbb{Q}$ | $\begin{pmatrix} 1 & \alpha_g^{(p)} \\ 0 & \lambda_g \end{pmatrix}$ | 2 | 1 | $1+uv$ | $\mathbb{Q}^* \ltimes \mathbb{Q}$ |
| $\zeta^{\mathfrak{m}}\begin{pmatrix} n \\ \xi \end{pmatrix}$ | $\mathbb{Q}^*\zeta^{\mathfrak{m}}\begin{pmatrix} n \\ \xi \end{pmatrix}+\mathbb{Q}$ | $\begin{pmatrix} 1 & \beta_g^{(n)} \\ 0 & \lambda_g^n \end{pmatrix}$ | 2 | 1 | $1+(uv)^n$ | $\mathbb{Q}^* \ltimes \mathbb{Q}$ |
| $\zeta^{\mathfrak{m}}\begin{pmatrix} 1,3 \\ -1,-1 \end{pmatrix}$ | $\mathbb{Q}^*\zeta^{\mathfrak{m}}\begin{pmatrix} 1,3 \\ -1,-1 \end{pmatrix}+\mathbb{Q}\zeta^{\mathfrak{m}}\begin{pmatrix} 3 \\ -1 \end{pmatrix}+\mathbb{Q}$ | $\begin{pmatrix} 1 & \beta_g^{(3)} & \beta_g^{(1,\overline{3})} \\ 0 & \lambda_g^3 & \lambda_g^3\beta_g^{(\overline{1})} \\ 0 & 0 & \lambda_g^4 \end{pmatrix}$ | 3 | 2 | $1+(uv)^3+(uv)^4$ | |

Here $\xi$ is a primitive N root of unity, $n \neq 1$, $N > 2$ [II].

`Reduced coaction` : $\Delta'(\zeta^{\mathfrak{m}}\begin{pmatrix} 1,3 \\ -1,-1 \end{pmatrix}) = \zeta^{\mathfrak{a}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \zeta^{\mathfrak{m}}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\Delta'(\zeta^{\mathfrak{m}}\begin{pmatrix} 1,3 \\ 1,-1 \end{pmatrix}) = \frac{7}{3}\zeta^{\mathfrak{a}}\begin{pmatrix} 3 \\ -1 \end{pmatrix} \otimes \zeta^{\mathfrak{m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$g.MP = MP.repr$, $\lambda_g \in \mathbb{Q}^*$, other in $\mathbb{Q}$.

`Remarks`:

- A unipotency degree 0: periods of pure motives; unipotency degree 1: periods of simple extensions.
  Unipotency degree k if $L^k U$ acts trivially, with L the lower central series.
- The rank, dimension of the representation associated.[III]
- Could also associate other invariants, like Hodge numbers (here type $(n,n)$ for CMMZV weight n) or even a Hodge polynomial.
  $\rightsquigarrow$ Can classify motivic periods via representation theoretic properties.

**Galois descent** : The idea is to look at, for $N' \mid N$:

`Question 1`: **How motivic periods of** $\mathcal{MT}_{N'}$ embeds into periods of $\mathcal{MT}_N$.

$\leftrightsquigarrow$ explicit CNS on the infinitesimal derivations $D_r$

(modulo smaller depth if there is a change of field)

`Question 2`: When a $\text{MZV}_{\mu_N} \in Vect_{\mathbb{Q}}\langle MZV_{\mu_{N'}} \rangle$.

NB: Because there is not always isomorphism between $H_N$ and $H_{MT_N}$, not always equivalent.[IV]

`The picture:`

$$(\mathcal{H}_{\mathcal{MT}_N})^{\mathcal{G}^{N/N'}} = \mathcal{H}_{\mathcal{MT}_{N'}}$$

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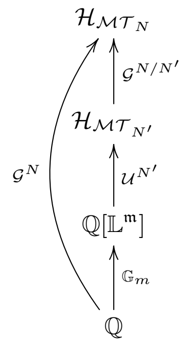
[I] Largest quotient of G which acts faithfully on $M(p)$

[II] For $N = 1, 2$ has to distinguish even or odd weights

[III] For $\alpha$ algebraic, rank is the dimension of the vector space spanned by conjugates, ie the degree of minimal annihilating polynmial.

[IV] Galois descent applies to $H^{MT_N}$. For $N' = 1, 2$, $i\pi^{\mathfrak{m}}$ has to be replaced by $\zeta^{\mathfrak{m}}(2)$, since only periods inv by Frob $\mathcal{F}_\infty$ in $\mathcal{H}^{N'}$.




`Example`: Case of the descent between MES and MMZV, the criteria depends only on weight 1 graded of the coaction:

$$\text{Z a MES is a Q CL MZV } \Leftrightarrow D_1(Z) \text{ and } \forall r \neq 1, D_r(Z) \text{ MMZV}$$

`Ex`: For $\zeta^{\mathfrak{m}}\begin{pmatrix} 3,3 \\ -1,-1 \end{pmatrix}$, we can check that $D_1(\bullet) = 0$, $D_5(\bullet) = 0$, which implies $\zeta^{\mathfrak{m}}\begin{pmatrix} 3,3 \\ -1,-1 \end{pmatrix} \in \mathcal{H}^1$.

`What else?`:

- Coming from category above to reach category underneath: can bring new basis of $H_{MT_{N'}}$ in terms of motivic periods in $H_{MT_N}$, or correct a term in $H_{MT_N}$ in order it lies in the category underneath $H_{MT_{N'}}$.
  `Ex`: Basis of MMZV with Deligne basis for MES.
  $\zeta\begin{pmatrix} 3,3 \\ 1,-1 \end{pmatrix} - 6\zeta\begin{pmatrix} 1,5 \\ 1,-1 \end{pmatrix} \in \mathcal{H}_1$.
- `Finding "missing" periods`: As said above, for N power of prime, $p > 3$, the fundamental groupoid does not generate the category, but using these Galois descents is a way to recover some or all missing periods.
  `Example`: for $p = 5$, can reach motivic periods in $\mathcal{MZ}_5$ which are not $MMZV_{\mu_5}$ by coming from the cat $\mathcal{MZ}_{10}$ (i.e. with $MMZV_{\mu_{10}}$).
- Also enables to `reach "new" motivic period` spaces, not known to be associated to a fundamental groupoid, and did not knew the motivic periods explicitely.
  `Example`: Basis of $\mathcal{H}_n^{\mathcal{MT}(\mathbb{Z}[\frac{1}{3}])}$ in terms of $MZV_{\mu_3}$.
- Can define `Higher ramification spaces` $\mathcal{F}_i\mathcal{H}_N$ (increasing motivic filtration) corresponding to generalized Galois descents.