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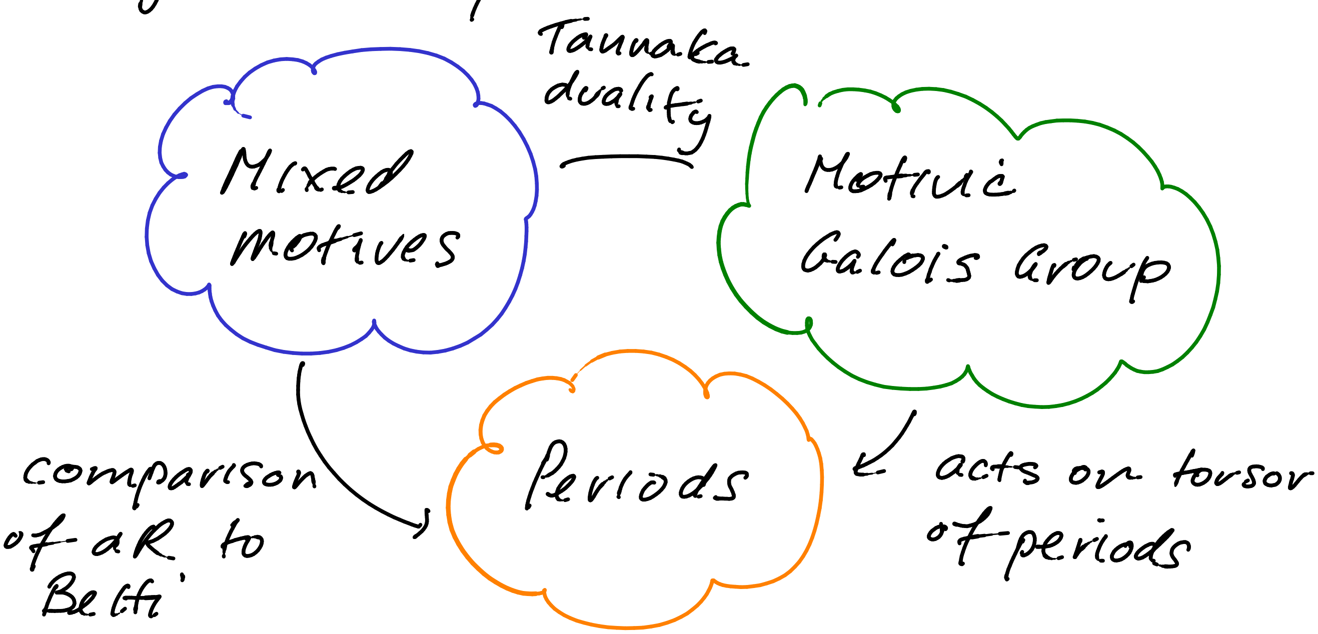
Motivic Galois Groups following Ayoub and Nori

Refs:

- Y. André : Bourbaki talk
- J. Ayoub : expository notes
- A. Huber & S. Müller-Stach: book.

⓪ Introduction :

Conjectural picture:



Assume $k \subseteq \mathbb{C}$. Candidates below.

(a) Nori motives:

$$NM(k) \overset{\text{Tannaka}}{\longleftrightarrow} G_N(k)$$

Nori motives (tannakian) ↗ $NM(k)$

$NM(k)$ —comp→ formal periods ← acts— $G_N(k)$

formal periods

↓ period conjecture

periods

Don't understand the Ext groups or relation to K-theory

(b) Voevodsky motives:

$DM_{gm}(k)$ — this would be the derived category of $MM(k)$

triangulated ⊗ category (correct exts) ↗ $DM_{gm}(k)$

$$DM_{gm}(k) \overset{\text{Tannaka}}{\longleftrightarrow} ?$$

$DM_{gm}(k)$ —?→ ? ←?— ?

?

Attempts to get tannakian cat of motives from $DM_{gm}(k)$

(1) Ayoub 1: 1-cats

(2) Ayoub 2: model categories

(3) Iwanari: stable ∞ categories

(4) Pridham: d.g. categories

② <u>Weak</u> <u>Tannakian</u> <u>formalism</u> (Ayoub)

triang $\otimes$ cat — tannakian cat

$$f : \mathcal{M} \longrightarrow \mathcal{C} \qquad \otimes \text{ functor}$$

$$\mathcal{M} \dashrightarrow T(\mathcal{M}, f) \nearrow \mathcal{C}$$

universal construction

$T(M, f)$ tannakian

$co\,Mod(H)$, H a commutative Hopf algebra in $\mathcal{C}$

e.g. A (left) comodule over

$$H = (h, \mu, \eta, \delta, \varepsilon)$$

is $\quad C = (c, \; c \to h \otimes c)$

Exercise: $coMod(H)$ is a symmetric monoidal category.

Recall: in classical Tannakian formalism

(1) the tannaka dual (coord ring) is typically large (eg ind-object)

(2) • mult, unit, counit "easy"

• comult, antipode "hard"

(A) If we assume $\mathcal{M} = \text{coMod}(H)$ comm Hopf alg in $\mathcal{C}$, then $\mathcal{M} \to \mathcal{C}$ will have a right adjoint $g = (H \otimes -)$

Assume $f : \mathcal{M} \to \mathcal{C}$ has a right adjoint g

$\Rightarrow$ • $fg\mathbb{1}$ commutative algebra

• $fg\mathbb{1} \to \mathbb{1}$ counit

(B) Assume

• f admits a monoidal section e

• e has a right-adjoint

• f & g satisfy a projection formula

$$gc \otimes M \xrightarrow{\sim} g(c \otimes fM)$$

<u>FACT</u>: If we assume A & B, then

(1) $fg\mathbb{1} =: H$ is a comm Hopf alg

(2) Have factoriz^n

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathcal{C} \\ & \tilde{f} \searrow & \uparrow \\ & & \mathrm{coMod}(H) \end{array}$$

H coacts trivially on image of e

(3) universal for (1) and (2)

Example: $\mathcal{G}$ pro-alg gp / $\mathbb{Q}$

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}}(\mathcal{G}) & \xrightarrow{f} & \mathrm{Mod}_{\mathbb{Q}} \\ & \searrow \quad \nearrow & \\ & \infty\text{-dim'l vect space} & \end{array}$$

recover $\mathcal{O}(\mathcal{G})$

Example:

$$DM(k) \xrightarrow{Re_B} D(\mathbb{Q})$$ Betti realization

X variety $/k \rightsquigarrow M(X) \in DM(k)$

$$M(X \times Y) = M(X) \otimes M(Y)$$

$$M(\mathbb{P}^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2]$$

$\mathbb{Q}(1)$ $\otimes$ invertible, dual $\mathbb{Q}(-1)$

$DM(k)$ triangulated $\otimes$ cat, $\mathbb{Q}$-linear

compactly generated by

$$M(X)(n) := M(X) \otimes \mathbb{Q}(n) \quad n \in \mathbb{Z}$$

<u>Relations:</u>
- $M(\mathbb{A}^1) = M(pt)$
- Mayer–Vietoris

$$DM_{gm}(k) \subseteq DM(k)$$

subcategory of compact (ie strongly dualizable) objects.

(eg: $\bigoplus_{n \geq 0} \mathbb{Q}(n)$ not dualizable)

For X smooth $/ k$ have

$$Re_B : M(X) \to S_*(X(\mathbb{C}), \mathbb{Q})$$

Assumptions A & B are satisfied automatically.

$$\rightsquigarrow \quad DM(k) \xrightarrow{Re_B} D(\mathbb{Q})$$

$$\text{universal} \searrow \qquad \nearrow$$

$$\text{coMod}(H_A(k))$$

$A \equiv$ Ayoub

③ <u>Connectivity</u> of $H_A(k)$ (Ayoub)

Would like to do

$\mathrm{Spec}(H_0(H_A(k)))$ motivic Galois group

Goal: $H_{<0}(H_A(k)) = 0$

(1) Grothendieck's comparison isom.

- $\Sigma \in DM(k)$ representing Betti cohomology
  $\Sigma = g\mathbb{1}$

- $\Omega \in DM(k,k)$ representing DR coho

To prove $H_A(k) = \mathrm{Re}_B \Sigma$ is connective, it suffices to prove $\mathrm{Re}_B(\Omega)$ is connective.

(2) Algebraic approximation of singular homology

$r > 1$ $\quad \mathbb{D}^n(r) \subseteq \mathbb{C}^n$ $\quad \mathbb{D}(r)$ radius $r > 1$.

Have

$\varprojlim_{r>1} \mathbb{D}^n(r)$ pro-analytic manifold

$\rightsquigarrow$ pro-variety $\overline{\mathbb{D}}^n_{\acute{e}t}$ étale / $\mathbb{A}^n$

Fact X smooth $/k$,

$S_g^{\overline{\mathbb{D}}^\bullet_{\acute{e}t}}(X)$ computes Betti homology

Assembling (1) + (2) gives

$$\mathcal{O}(\overline{\mathbb{D}}^n_{\acute{e}t}) = \left\{ \begin{array}{l} f \text{ analytic on } \overline{\mathbb{D}}^n \\ \text{algebraic over} \\ \quad k(z_1, \dots, z_n) \end{array} \right\}$$

$$\| $$

$$\mathcal{O}_{alg}(\overline{\mathbb{D}}^n)$$

$\Rightarrow$ $R_B^{eff}(\Omega^{eff})$ is quasi-iso to

$$\xrightarrow{d} \Omega^{\infty-n} \to \Omega^{\infty-(n-1)} \to \cdots$$

$$\to \Omega^{\infty-1} \to \Omega^{\infty} \to 0$$

$$[0] \qquad [-1]$$

$$\Omega^{\infty-n} = \bigoplus_{\substack{I \subseteq \mathbb{N}_+ \\ |I| = n}} \mathcal{O}_{alg}^{(I)} (\overline{\mathbb{D}}^{\infty}) \, d\hat{z}_I$$

def

$$\left\{ f : f \text{ vanishes on } \prod_{i \in I} z_i \cdot (z_i - 1) \right\}$$

Upshot: Get:

(1) $\mathcal{G}_A(k) = \operatorname{Spec} H_0(H_A \cdot k))$

(2) tannakian cat $\operatorname{Rep}_{\mathbb{Q}} \mathcal{G}_A(k)$

FACT: (Choudhary – Gallauer)

$$\mathcal{G}_A(k) \cong \mathcal{G}_N(k) \quad \text{Nori's group.}$$

(4) Application to Periods

K-Z-

Kontsevich–Zagier

$$\wp^{eff}_{KZ}(k) = \frac{\left\{ \begin{array}{l} f \text{ analytic on some } \overline{\mathbb{D}}^n \\ \text{algebraic over } k(z_1, \ldots, z_n) \end{array} \right\}}{\langle \partial f / \partial z_i - f|_{z_i=0} - f|_{z_i=1} \rangle}$$

How to get a period integral from this:

(X, Z, ω, γ) formal period

$X = \mathrm{Spec}(A)$

A étale over $k[z_1, \dots, z_n]$

$Z = \prod z_i (z_i - 1)$

$\omega = f \, dz_1 \wedge \dots \wedge dz_n$

$\gamma : [0,1]^n \longrightarrow \overline{\mathbb{D}}^n \longrightarrow X(\mathbb{C})$

$$f \longmapsto \int_{[0,1]^n} f \quad , \qquad f \in A.$$